A GENERAL THEORY OF FINITE ELEMENTS
II. APPLICATIONS

J. TINSLEY ODEN

Professor of Engineering Mechanics, Research Institute,
University of Alabama in Huntsville, Huntsville, Alabama

SUMMARY

In Part I of the this paper, topological properties of finite element models of functions defined on spaces of finite dimension were examined. In this part, a number of applications of the general theory are presented. These include the generation of finite element models in the time domain and certain problems in wave propagation, kinetic theory of gases, non-linear partial differential equations, non-linear continuum mechanics, and fluid dynamics.

INTRODUCTION

In Part I, the following observations were made: Let $T(X)$ denote the value of a continuous function at a point X in a k-dimensional space $\mathscr{E}^k$ and its values are arbitrary in that they may be scalars, vectors, tensors of any order, etc. The region $\mathscr{R}$ can be replaced by a region $\bar{\mathscr{R}}$ containing a finite number G of nodal points X_Δ in $\mathscr{R}$ or by a set $\mathscr{R}^*$ consisting of a collection of E disjoint subregions r_e called finite elements. The process of connecting the elements together is accomplished by a singular mapping $\Omega \colon \mathscr{R}_x \to \mathscr{R}_x^*$ which maps global nodal points X_Δ into appropriate local points $x_N^{(e)}$ in the connected model. Since $T(X)$ is one-to-one on $\mathscr{R}$, a similar procedure applies to the finite element model of $T(X)$. In fact, if $t^{(e)}(x)$ is the local field associated with element e and $t_N^{(e)}$ are its values at node N of the element, then

$$t_N^{(e)} = \overset{(e)}{\Omega_N^\Delta} T_\Delta \tag{1}$$

where $\overset{(e)}{\Omega_N^\Delta}$ is defined in Part I, equation 4, and $T_\Delta = T(X_\Delta)$, $\Delta = 1, 2, \ldots, G$. The local fields are approximated over each element by

$$t^{(e)}(x) \approx \Psi_N^{(e)}(x) t_N^{(e)} \tag{2}$$

where the normalized interpolating (Lagrange) functions are selected so that (a) $\Psi_N^{(e)}(x_M) = \delta_{MN}$; $M, N = 1, 2, \ldots, N_e$, where N_e is the number of nodes belonging to element e, and (b) the finite element representation of $T(X)$ is continuous across interelement boundaries in the connected model. The final form of the (first-order) finite element representation of $T(X)$ is then

$$\bar{T}(x) = \sum_{e=1}^{E} \Psi_N^{(e)}(x) \overset{(e)}{\Omega_N^\Delta} T_\Delta \tag{3}$$

To apply the concepts presented previously to any type of linear or non-linear field problem, all that is needed is some means to translate a relation that holds at a point into one that must hold

over a finite region. In solving partial differential equations, Zienkiewicz and Cheung, Part I, Reference 4, have shown that this translation from point relations to regional relations is best provided by equivalent variational statements of the problem. In problems of physics, it may also be provided by local and global forms of the balance laws of thermodynamics and electro-magnetics. The possibility of applying the general equations to cases in which the independent variables are something other than the usual spatial co-ordinates is also interesting. In the following, we consider several examples.

FINITE ELEMENTS IN THE TIME DOMAIN

Since the finite element models described previously can, in principle, be used to approximate functions defined on spaces of any finite dimension, it is natural to first question their utility in representing functions in the four-dimensional space–time domain.

Consider, for example, a scalar-valued function $\Phi\ (X^1,\ X^2,\ X^3,\ t)$ of position X^i and time t. Clearly, we can set $X^4 = t$ and consider $\Phi\ (\mathbf{X})$ to be defined on a four-dimensional space, the points of which are given by $X^i(i = 1,2,3,4)$. If the domain is represented by a collection of E four-dimensional sub-domains, then we can concentrate on approximating the field $\phi^{(e)}(x^1,\ x^2,\ x^3,\ x^4)$ locally over a typical finite element e:

$$\phi^{(e)} = \Psi_N(\mathbf{x})\phi_N^{(e)} \tag{4}$$

where the interpolating functions $\Psi_N(\mathbf{x})$ are functions of $x^1,\ x^2,\ x^3,\ x^4$. The general process of assembling elements amounts to a simple application of equation (1) and is identical to that described for general spaces in Part I of this paper. However, the details of applying the model to propagation problems, hyperbolic and parabolic partial differential equations, etc. can only be appreciated through specific examples.

Two-dimensional space-time

To illustrate the use of finite elements in the time domain, consider the simple example of a one-dimensional elastic bar of length L and cross-sectional area A, subjected to either a prescribed force $P(t)$ at a free end or to prescribed initial displacement $u(x,0) = f(x)$. In con-ventional finite element models, the longitudinal displacement $u = u(x,t)$ is approximated by one-dimensional interpolating functions $\Psi_N(x)$ $(N = 1,2,\ldots,N_e)$ multiplied by nodal displace-ments which are functions of time. In the present analysis, the displacements $u_N^{(e)}$ are values of a local field $u^{(e)}(x,t)$ defined over a region in a two-dimensional space (x,t) and are independent of x and t. The interpolating functions are functions of both the longitudinal co-ordinate x and time t: $\Psi_N = \Psi_N(x,t)$. Thus, for a typical finite element

$$u^{(e)}(x,t) = \Psi_N(x,t)u_N^{(e)} \tag{5}$$

The Lagrangian potential $\mathscr{L}$ for a linearly elastic bar of modulus E and mass density ρ is

$$\mathscr{L} = \frac{1}{2}\iint_{t\ v}\left[\rho\left(\frac{\partial u}{\partial t}\right)^2 - E\left(\frac{\partial u}{\partial x}\right)^2\right]\mathrm{d}v\,\mathrm{d}t + \int_t\sum_\alpha S_\alpha(t)u(x_\alpha,t)\,\mathrm{d}t \tag{6}$$

where v is the volume of the bar and $-\sum_\alpha S_\alpha(t)u_\alpha(x_\alpha,t) = -S_1(t)u(x_1,t) - S_1(t)u(x_2,t)$ is the poten-tial energy of forces $S_1(t)$ and $S_2(t)$ at the ends of the segment $[x_1,x_2]$ of the bar under consideration. Thus, for a typical finite element e,

$$\mathscr{L}_e = \tfrac{1}{2}a_{MN}^{(e)}u_M^{(e)}u_N^{(e)} + p_N^{(e)}u_N^{(e)} \tag{7}$$

wherein

$$a_{MN}^e = \iint\limits_{t\ v}\left(\rho\frac{\partial\Psi_M}{\partial t}\frac{\partial\Psi_N}{\partial t} - E\frac{\partial\Psi_M}{\partial x}\frac{\partial\Psi_N}{\partial x}\right)dv\,dt \tag{8}$$

and

$$p_N^{(e)} = \int_t\sum_\alpha S_\alpha(t)u(x_\alpha,t)\,dt \tag{9}$$

In these equations, the integration is taken over the portion of the time domain spanned by the element.

In view of equation (7), the Lagrangian $\mathscr{L}$ has an interesting property that differs significantly from the usual case: it is not a functional of velocity. Indeed, $\mathscr{L}$ becomes an ordinary function of nodal values of displacements; but because of the particular type of formulation, these are independent of time. Hamilton's principle, of course, still applies so that

$$\delta\mathscr{L}_{(e)} = \frac{\partial\mathscr{L}_{(e)}}{\partial u_N^{(e)}}\delta u_N^{(e)} = 0 \tag{10}$$

and we obtain

$$a_{MN}^{(e)}u_M^{(e)} + p_N^{(e)} = 0 \tag{11}$$

The process of assembling the elements into the total model follows the usual procedure for conventional two-dimensional finite-element models.

Longitudinal waves

It is important to note that the procedure by which the above finite element equations are solved is quite different than for purely elliptic-type problems. In fact, equation (11) is the finite element analogue of the hyperbolic wave equation

$$\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial^2 u}{\partial t^2} = f(t) \tag{12}$$

where $\alpha = \sqrt{E/\rho}$.

To illustrate the procedure, consider the simple example in which the local field is given by the linear approximation

$$u^{(e)} = \Psi_N(x,t)u_N^{(e)} = a + bx + ct \tag{13}$$

where a, b, and c are constants and $N = 1,2,3$. In this case the finite element is a triangle in two-dimensional space-time, such as is indicated in Figure 1. From equations (42) and (45) in Part I of this paper we find that

$$\left.\begin{aligned}
\Psi_1(x,t) &= \frac{1}{2\Delta}[(x_2 t_3 - x_3 t_2) + (t_2 - t_3)x + (x_3 - x_2)t]\\[4pt]
\Psi_2(x,t) &= \frac{1}{2\Delta}[(x_3 t_1 - x_1 t_3) + (t_3 - t_1)x + (x_1 - x_3)t]\\[4pt]
\Psi_3(x,t) &= \frac{1}{2\Delta}[(x_1 t_2 - x_2 t_1) + (t_2 - t_1)x + (x_2 - x_1)t]
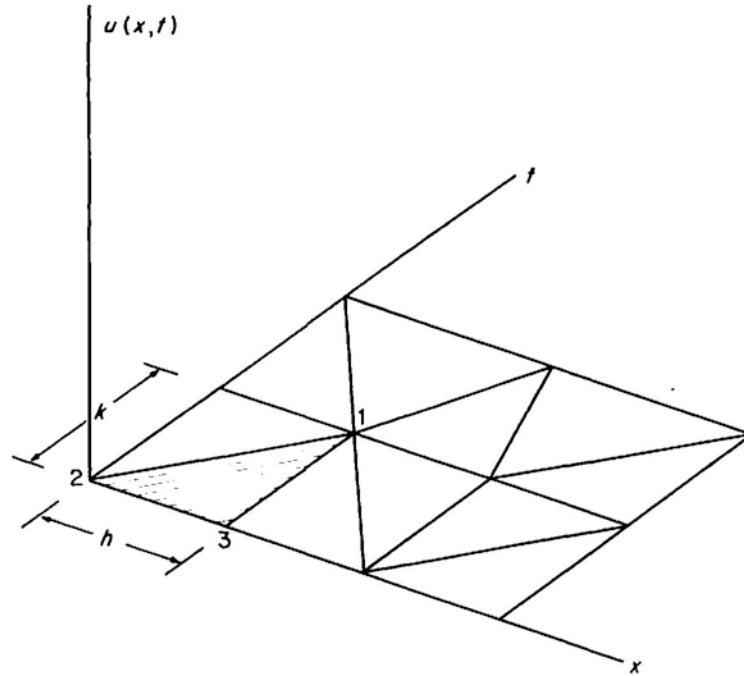\end{aligned}\right\} \tag{14}$$

Figure 1. Finite elements in the time domain

where Δ is the area of the element in the x,t-plane. For example, introducing the geometry of shaded element in Figure 1 into equation (14) and making use of equation (11), we find that for this rather crude approximation the local equations take the form

$$\left.\begin{array}{l} \hat{p}_1^{(e)} = u_1^{(e)} - u_3^{(e)} \\ \hat{p}_2^{(e)} = \lambda^2(u_3^{(e)} - u_1^{(e)}) \\ \hat{p}_3^{(e)} = u_3^{(e)} - u_1^{(e)} - \lambda^2(u_3^{(e)} - u_1^{(e)}) \end{array}\right\} \tag{15}$$

in which $\hat{p}_N^{(e)} = -k^2 p_N^{(e)}/A\rho\Delta$ and $\lambda^2 = k^2\alpha^2/h^2$

Suppose that $u(x,0) = f(x)$, $u(0,t) = u(L,t) = 0$, and $\partial u(x,0)/\partial t = 0$ are the given boundary and initial conditions and that the finite-element network shown in Figure 2 is used. The analysis proceeds as follows:

1. Conceptually, only one tier of elements (the first row corresponding to the interval $0 \le t \le k$, the second $k \le t \le 2k$, etc.) need be considered to be generated at a time. Global values U_Δ of the displacements of boundary nodes are equated to zero in agreement with given boundary conditions: $U_1 = U_6 (=U_{11} = U_{16} = \ldots) = 0$, $U_5 = U_{10} (= U_{15} = U_{20} = \ldots) = 0$. Displacements at interior nodes corresponding to $t = 0$ take on the prescribed values; i.e. $U_2 = f(h)$, $U_3 = f(2h)$, $U_4 = f(3h)$, etc.

2. Since the displacements U_2, U_3, U_3 take on prescribed values, the corresponding global generalized 'forces' (conjugate variables) $\hat{P}_2$, $\hat{P}_3$, $\hat{P}_4$ vanish. The only unknowns in the resulting equations

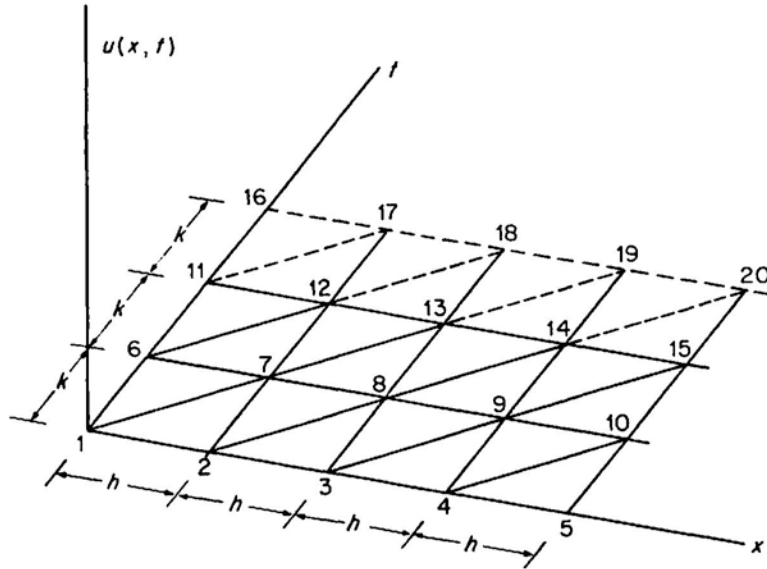$$\hat{P}_2(U_2,U_7) = \hat{P}_3(U_3,U_8) = \hat{P}_4(U_4,U_9) = 0 \tag{16}$$

Figure 2. Example of propagation of solution in time

are the nodal values U_7, U_8, U_9 which represent the displaced profile after k seconds. Since each equation in (16) has only one unknown, the set can be solved immediately to give U_7, U_8, and U_9 in terms of the prescribed nodal displacements at $t = 0$.

3. Another tier of elements ($k \leq t \leq 2k$) is now considered. Displacements U_{12}, U_{13}, and U_{14} are obtained from the conditions $\hat{P}_7 = \hat{P}_8 = \hat{P}_9 = 0$. Then a third tier of elements is considered and the process is repeated.

Thus the finite element solution is propagated in time in a manner similar to conventional finite difference solutions.

We remark that in the case in which a time-varying end load is applied and initial displacements $u(x,o)$ are not prescribed, the same procedure is followed except that U_5, U_{10}, ... $\neq 0$ and, instead of equation (16), $\hat{P}_2$, $\hat{P}_3$, $\hat{P}_4$ (and $\hat{P}_7$, $\hat{P}_8$, ... etc.) take on prescribed values.

Stability

The rather crude simplex model used in the above example is the most primitive finite model for the problem at hand. By using higher-order approximations or adding more degrees of freedom to the elements, much greater accuracy, more stable solutions, and smoother results can be obtained for more difficult propagation problems. Nevertheless, it is interesting to note that for an interior node such as 8 in the mesh indicated in Figure 2 we have

$$-2\hat{P}_8 = 0 = -(U_3 - 2U_8 + U_{13}) + \lambda^2(U_7 - 2U_8 + U_9) \qquad (17)$$

which is precisely the form of the first-order central difference approximation of equation (12). Thus, we can draw on the Courant, Friedrichs, Lewy criteria[2] to obtain conclusions on the stability of the scheme outlined above. Accordingly, the solution is unstable for $\lambda > 1$ and violently unstable for increasing values of λ; for $\lambda < 1$ it is stable but the accuracy decreases with decreasing λ; for $\lambda = 1$, the solution is stable and agrees with the exact solution of equation (12).

FINITE ELEMENTS IN THE COMPLEX PLANE: SCHROEDINGER'S EQUATION

In quantum mechanics, the Schroedinger wave equation can be written in terms of a wave function χ which is complex. We now examine the development of finite element analogues of Schroedinger's equations for χ and its complex conjugate $\bar{\chi}$ for the case of a single particle of mass m acting under the influence of a potential field $V(\mathbf{x}) = V(x,y,z)$.

The wave function $\chi(x,t)$ can be written in the form

$$\chi(\mathbf{x},t) = u(\mathbf{x},t) + iv(\mathbf{x},t) \tag{18}$$

where $i = \sqrt{-1}$. The complex conjugate is $\bar{\chi}(\mathbf{x},t) = u(\mathbf{x},t) - iv(\mathbf{x},t)$ and physically $\chi(\mathbf{x},t)\bar{\chi}(\mathbf{x},t)$ represents the probability density at time t for the presence of the particle for the configuration of the system specified by the co-ordinates $\mathbf{x}$. Confining our attention to a typical finite element e, we approximate the real and imaginary parts of $\chi(x,t)$ locally by

$$u^{(e)}(\mathbf{x},t) = \Psi_N(\mathbf{x})u_N^{(e)} \quad , \quad v^{(e)}(\mathbf{x},t) = \Psi_N(\mathbf{x})v_N^{(e)} \tag{19a,b}$$

where $u_N^{(e)}$, $v_N^{(e)}$ are the time-dependent nodal values of $u(\mathbf{x},t)$ and $v(\mathbf{x},t)$. Then

$$\chi^{(e)}(\mathbf{x},t) = \Psi_N(\mathbf{x})\chi_N^{(e)} \tag{20a}$$

$$\bar{\chi}^{(e)}(\mathbf{x},t) = \Psi_N(\mathbf{x})\bar{\chi}_N^{(e)} \tag{20b}$$

where

$$\chi_N^{(e)} = u_N^{(e)} + iv_N^{(e)} \quad , \quad \bar{\chi}_N^{(e)} = u_N^{(e)} - iv_N^{(e)} \tag{21a,b}$$

The Lagrange density $L^{(e)}$ for an element is [3]

$$L^{(e)} = \frac{h^2}{8\pi^2 m}\,\text{grad}\,\bar{\chi} \cdot \text{grad}\,\chi - \frac{h}{4\pi i}\left(\bar{\chi}\frac{\partial \chi}{\partial t} - \frac{\partial \bar{\chi}}{\partial t}\chi\right) - \bar{\chi}V\bar{\chi} \tag{22}$$

where h is Planck's constant and $\bar{\chi}$ and χ are to be varied independently until $\mathscr{L} = \iiiint L^{(e)}\,d\mathscr{R}\,dt$ is a minimum. Introducing (20) into (22) and requiring that

$$\frac{d}{dt}\left(\frac{\partial L^{(e)}}{\partial \dot{\chi}_N}\right) - \frac{\partial L}{\partial \chi_N} = \frac{d}{dt}\left(\frac{\partial L^{(e)}}{\partial \dot{\bar{\chi}}_N}\right) - \frac{\partial L^{(e)}}{\partial \chi_N} = 0 \tag{23}$$

we arrive at the pair of equations

$$\frac{h^2}{8\pi^2 m}\alpha_{MN}^{(e)}\bar{\chi}_M^{(e)} + \frac{h}{4\pi i}\beta_{MN}^{(e)}\dot{\bar{\chi}}_M^{(e)} - \gamma_{MN}^{(e)}\bar{\chi}_M^{(e)} = 0 \tag{24a}$$

$$\frac{h^2}{8\pi^2 m}\alpha_{MN}^{(e)}\chi_M^{(e)} - \frac{h}{4\pi i}\beta_{MN}^{(e)}\dot{\chi}_M^{(e)} - \gamma_{MN}^{(e)}\chi_M^{(e)} = 0 \tag{24b}$$

wherein

$$\alpha_{MN}^{(e)} = \int_{\mathscr{R}} \Psi_{M,i}(\mathbf{x})\Psi_{N,i}(\mathbf{x})\,d\mathscr{R} \tag{25a}$$

$$\beta_{MN}^{(e)} = \int_{\mathscr{R}} \Psi(\mathbf{x})\Psi(\mathbf{x})\,d\mathscr{R} \tag{25b}$$

and

$$\gamma_{MN}^{(e)} = \int_{\mathscr{R}} \Psi_M(\mathbf{x})V(\mathbf{x})\Psi_N(\mathbf{x})\,d\mathscr{R} \tag{25c}$$

Equations (24) are the discrete counterparts of the Schroedinger wave equations for the finite element. The quantities $h\beta^{(e)}_{MN}\chi^{(e)}_M/4\pi i$, $h\beta^{(e)}_{MN}\bar{\chi}^{(e)}_M/4\pi i$ are the generalized canonical momenta at note N of the element, $(h^2\alpha^{(e)}_{MN}/8\pi^2 m) - \gamma^{(e)}_{MN}$ represents the discrete equivalent of the Hamiltonian operator for the particle while in finite element e.

KINETIC THEORY OF GASES

The statistical mechanics of dilute gases involves problem areas in which finite element models in the six-dimensional μ-space may be used to advantage. Here the molecular density is assumed to be sufficiently low and the temperature sufficiently high that each molecule of gas can be considered to be a classical particle with a reasonably well-defined position and momentum. The behaviour of a contained gas is characterized, according to classical kinetic theory,[4] by a distribution function $f(x, v, t)$ which is defined so as to represent the number of molecules at time t which have positions lying in a 'volume' element $d\Omega$ in a six-dimensional velocity-space, such that x_1, x_2, and x_3 denote the position of the molecule and $x_4 = v_1$, $x_5 = v_2$, $x_6 = v_6$ its components of velocity. Unlike classical mechanics, which deals only with mean velocities, the quantities v_1, v_2 and v_3 are independent of x_1, x_2, and x_3. We outline briefly the finite element approximation of such distribution functions.

Following the procedures outlined in Part I, can immediately write down the local approximation of the distribution function over an element e in six-dimensional space:

$$f^{(e)}(x, t) \approx \Psi_N(x_1, x_2, \ldots, x_6) f^{(e)}_N \tag{26}$$

where $f^{(e)}_N$ are functions of time and $N = 1, 2, \ldots, N_e$. As a first approximation, we may, for example, use the simplex approximation wherein the interpolating functions $\Psi_N(x)$ are of the form

$$\Psi_N(x) = a_N + b_{Ni}x^i \qquad i = 1, 2, \ldots, 6 \tag{27}$$

where $N_e = 7$ and a_N, b_{Ni} can be expressed in terms of the nodal 'co-ordinates' x^i_N.

A 'volume' element in six-dimensional μ-space is denoted $d\Omega = d\mathscr{R}\, dV$ where, for simplicity, we may take $d\mathscr{R} = dx_1\, dx_2\, dx_3$ to be the usual three-dimensional volume element and $dV = dx_4\, dx_5\, dx_6$ a volume in velocity space about v. Then $f^{(e)}(x, t)\, d\Omega$ is the number of molecules in $d\Omega$ at time t at a point x in finite element e. For every dilute gases at high temperatures, $f^{(e)}(x, t)$ obeys the collisionless transport equation

$$\frac{\partial f^{(e)}}{\partial t} + v \cdot \nabla_r f^{(e)} = 0 \tag{28}$$

where ∇_r is the gradient operator with respect to $r = (x_1, x_2, x_3)$. Multiplying equation (28) through by $f^e\, d\Omega$ and integrating over the volume Ω_e of the element, we obtain

$$(r^{(e)}_{MN}\dot{f}^{(e)}_M + k^{(e)}_{MN}f^{(e)}_M)f^{(e)}_N = 0 \tag{29}$$

where

$$r^{(e)}_{MN} = \int_{\Omega_e} \Psi_N(x)\Psi_M(x)\, d\Omega, \quad k^{(e)}_{MN} = \int_{\Omega_e} \Psi_N(x)\sum_{m=1}^{3} v_m \frac{\partial\Psi_M(x)}{\partial x_m}\, d\Omega \tag{30a,b}$$

Since (29) must hold for arbitrary $f^{(e)}_N$, we have for a typical element

$$r_{MN}^{(e)} \dot{f}_M^{(e)} + k_{MN}^{(e)} f_M^{(e)} = 0 \tag{31}$$

in which $M, N = 1, 2, \ldots, N_e$. The process of connecting elements is no different than that indicated by (1).

Equation (31) is the transport equation for a finite element. It represents the discrete counterpart of the collisionless Boltzmann–Vlasov transport equation. We observe that the procedure used to obtain the finite element model (31) is essentially a local application of Galerkin's method.

A NON-LINEAR PARTIAL DIFFERENTIAL EQUATION

In a recent paper, Greenspan[5] presented a general method for solving boundary-value problems for non-linear differential equations which involved using finite differences for approximating the functionals appearing in an associated variational statement of the problem. Zienkiewicz and Cheung,[1] [Part I, Reference 4] used a similar procedure for the finite element solution of a class of linear partial differential equations. It is a simple matter to extend these finite element procedures to solve non-linear partial differential equations.

As an example, consider the non-linear boundary-value problem which involves finding a solution $\phi(x^1, x^2)$, over a closed region $\mathcal{R}$ of two-dimensional Euclidean space, of the non-linear partial differential equation

$$2\phi\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 - f(x, y) = 0 \tag{32}$$

subject to the conditions on the boundary curve C:

$$\phi(s) = g(s) \text{ on C} \tag{33}$$

The associated variational problem involves finding an extremum of the functional

$$I = \frac{1}{2}\int_{\mathcal{R}}(\phi\phi_{,\alpha}\phi_{,\alpha} - f\phi)\,d\mathcal{R} - \oint_C \phi g(s)\,ds \tag{34}$$

where $\alpha = 1, 2$; $x^1 = x$ and $x^2 = y$; and $\phi_{,\alpha} \equiv \partial\phi/\partial x^\alpha$.

The local approximation of $\phi(x)$ over a finite element e of $\mathcal{R}$ is

$$\bar{\phi}^{(e)}(x) = \Psi_N(x)\phi_N^{(e)} \tag{35}$$

and for the disjoint element e, we have from

$$\begin{aligned}
I_e &= \frac{1}{2}\int_{\tau_e}\bar{\phi}^{(e)}(\bar{\phi}_{,\alpha}^{(e)}\bar{\phi}_{,\alpha}^{(e)} - f\bar{\phi}^{(e)})\,d\tau_e - \oint_{C_e}\bar{\phi}^{(e)}g(s)\,ds \\
&= \frac{1}{3}\,d_{(e)}^{NMS}\phi_N^{(e)}\phi_M^{(e)}\phi_S^{(e)} - p_{(e)}^S\phi_S^{(e)}
\end{aligned} \tag{36}$$

where

$$d_{(e)}^{NMS} = \frac{3}{2}\int_{\tau_e}\Psi_N(x)\Psi_{M,\alpha}(x)\Psi_{S,\alpha}(x)\,d\tau_e \tag{37}$$

$$p_{(e)}^S = \int_{\mathbf{r}_e} f(\mathbf{x})\Psi_S(x)\,\mathrm{d}\mathbf{r}_e + \oint_{C_e} \Psi_S(x)g(s)\,\mathrm{d}s = f_{(e)}^S + g_{(e)}^S \tag{38}$$

Thus, for the element,

$$\delta I_e = \frac{\partial I_e}{\partial \phi_N}\,\delta\phi_N = 0 \tag{39}$$

and we obtain the quadratic equations

$$\mathrm{d}_{(e)}^{NMS}\phi_N^{(e)}\phi_M^{(e)} - p_{(e)}^S = 0 \tag{40}$$

where $N, M, S = 1, 2, \ldots, N_e$

Final global equations are of the form

$$P^\Delta = D^{\Delta\Gamma\Omega}\Phi_\Gamma\Phi_\Omega \tag{41}$$

where

$$D^{\Delta\Gamma\Omega} = \sum_e^E \Omega_S^{(e)\Delta}\Omega_N^{(e)\Gamma}\Omega_M^{(e)\Omega}\,\mathrm{d}_{(e)}^{NMS} \tag{42}$$

and

$$\phi_N^{(e)} = \Omega_N^{(e)\Delta}\Phi_\Delta \qquad P^\Delta = \sum_e \Omega_S^{(e)\Delta}p_{(e)}^S \tag{43b,c}$$

Note that in equation (38) the global values $G^\Delta = \sum_e \Omega_S^{(e)\Delta}g_{(e)}^S$ of $g_{(e)}^S$ are zero at all interior nodes and they take on prescribed values on the boundary of $\mathscr{R}$ (or $\bar{\mathscr{R}}$). On applying such boundary conditions, the system of non-linear equations may be solved by (say) the Newton–Raphson Method.

NON-LINEAR CONTINUA

We now demonstrate briefly that the finite element method can be applied to *any* type of non-linear continua, whether it be dissipative or not, without going through the usual ritual of developing associated variational principles for the problem at hand. The global form of the law of conservation of energy for thermomechanical phenomena is taken to be[5]

$$\dot{\mathscr{V}} + \dot{U} = \Omega + Q \tag{44}$$

where $\mathscr{V}$ is the kinetic energy, U the internal energy, Ω the mechanical power, Q the rate of heat input, and the superposed dots indicate time rates:

$$\mathscr{V} = \tfrac{1}{2}\int_v \rho\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}v \tag{45a}$$

$$U = \int_v \rho\varepsilon\,\mathrm{d}v \tag{45b}$$

$$W = \int_v \rho\mathbf{F}\cdot\mathbf{v}\,\mathrm{d}v + \int_A \mathbf{S}\cdot\mathbf{v}\,\mathrm{d}A \tag{45c}$$

$$Q = \int_v \rho h\,\mathrm{d}v + \int_A \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}A \tag{45d}$$

Here $\mathbf{v}$ is the velocity, ρ the mass density, ε the internal energy density, $\mathbf{F}$ the body force per unit

mass, S the surface tractions, h the heat supplied from internal sources, q the heat flux, and n a unit vector normal to the surface area A.

If the motion is referred to an intrinsic co-ordinate system x^i ($i = 1, 2, 3$) which is rectangular in the reference configuration, we have the strain-displacement relations

$$2\gamma_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \tag{46}$$

and, if linear momentum is conserved, we have for the local form of equation (44)

$$\rho\dot{e} = \sigma^{ij}(\delta_{im} + u_{m,i})\dot{u}_{m,j} + q^k_{;k} + \rho h \tag{47}$$

where σ^{ij} are contravariant components of the stress tensor referred to x^i and semicolon indicates covariant differentiation with respect to the convected co-ordinates x^i. In addition, we need constitutive equations for the material, which, for illustration purposes, we will take to be of the form

$$\sigma^{ij} = \mathbb{C}^{ij}[\gamma_{ij}(\tau - t), \dot{\gamma}_{ij}(\tau - t) \dots] \tag{48}$$

Here $\mathbb{C}^{ij}$ are constitutive functionals of, perhaps, the histories of the strain, strain rates, higher-order strain rates, temperature, etc.

We now consider a typical finite element of the continuum. The displacement field associated with element e is

$$u_i^{(e)} = \Psi_N^{(e)}(x)u_{Ni}^{(e)} \tag{49}$$

where $u_{Ni}^{(e)}$ are the components of displacement of node N of element e. Introducing equations (47–49) into (45) and (44) and simplifying, we arrive at the general equation of energy balance for a finite element of a continuous media:

$$m_{NM}^{(e)}\ddot{u}_{Mi}^{(e)}\dot{u}_{Mi}^{(e)} + \int_{v_e} \sigma^{kj}(\delta_{ki} + \Psi_{R,k}^{(e)}u_{Ri}^{(e)})\Psi_{M,j}\dot{u}_{Mi}^{(e)} \, dv = p_{Mi}^{(e)}\dot{u}_{Mi}^{(e)} \tag{50}$$

where

$$m_{NM}^{(e)} = \int_{v_e} \Psi_N^{(e)}(x)\Psi_M^{(e)}(x) \, dv \tag{51a}$$

$$p_{Ni}^{(e)} = \int_{v_e} \rho F_i \Psi_N(x) \, dv + \int_A S_i \Psi_N(x) \, dA \tag{51b}$$

Here $m_{NM}^{(e)}$ is the consistent mass matrix for the continua and $p_{Ni}^{(e)}$ are the components of generalized force at node N of the element. Since (50) must hold for arbitrary nodal velocities, we have

$$m_{NM}^{(e)}\ddot{u}_{Mi}^{(e)} + \int_{v_e} \mathbb{C}^{kj}(\delta_{ki} + \Psi_{R,k}^{(e)}(x)u_{R,i}^{(e)}) \, dv = p_{Ni}^{(e)} \tag{52}$$

in which it is understood that $\mathbb{C}^{ij}$ has been put in terms of $u_{Ni}^{(e)}$, $u_{Ni}^{(e)}$, ..., etc. with the aid of (48) and (46).

Equation (52) is the general equation of motion of finite elements of non-linear continua. Specific forms of these equations can be obtained only after specific constitutive equations are introduced. Final equations for the assembled system of elements are obtained, as before, by using the transformation pair in equations (14) and (33) of Part I.

In the following section we examine different forms of equation (52) which are written from an Eulerian description of the motion. Then the procedure leads to finite element models of problems in fluid dynamics.

FLUID DYNAMICS

Although the results of the previous section are applicable to any type of continua, we now examine an alternate formulation of finite element models which is specifically designed for the general problem of dynamics of a continuous fluid. The type of fluid is arbitrary: it may be compressible, incompressible, inviscid, viscoelastic, etc., and, again, no use is made of specific variational principles in the sense that the formulation does not depend on the existence of extremal principles involving well-defined functionals. Thus, the classical problems of 'potential' flow* comprise only a special sub-class of those for which this formulation holds.

A fundamental difference between finite element modules of fluid motion and the motion of a solid is that in problems of fluid dynamics finite elements represent spatial rather than material sub-regions of the continuum. Thus, instead of representing finite elements of a fluid material, the elements represent sub-regions in the space through which the fluid moves (i.e. the Eulerian description of motion). Finite element models of velocity fields over an element are specified in terms velocities *at* nodes in space rather than *of* nodes.

Consider a typical finite element e of a region in $\mathscr{E}^k$ through which a fluid moves with (local) velocity $\mathbf{v}(\mathbf{x}) = \mathbf{v}(x_1, x_2, x_3, t)$. If $v_i^{(e)}$ denotes the Cartesian components of velocity in e, then in accordance with (1), the field is approximated over the finite element by

$$v_i^{(e)} = \Psi_N(\mathbf{x})v_{Ni}^{(e)} \tag{53}$$

in which $v_{Ni}^{(e)}$ $(N = 1,2,\ldots,N_e)$ are the values of the velocity components at node N of the element. The coordinates $\mathbf{x}$ now pertain to a point in the current configuration.

The Eulerian (spatial) form of the first law of thermodynamics is, ignoring thermal effects,

$$\int_{v_e} \rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right)v_i\,dv + \int_{v_e} \rho\dot{\varepsilon}\,dv = \int_{v_e} \rho F_i v_i\,dv + \int_{A_e} S_i v_i\,dA \tag{54}$$

where ρ is the mass density, ε the specific internal energy, F_i the components of body force, and S_i the components of surface traction. Locally,

$$\rho\dot{\varepsilon} = t_{ij}\,d_{ij} \tag{55}$$

where t_{ij} is the stress tensor referred to a spatial Cartesian frame and d_{ij} is the deformation rate tensor:

$$2d_{ij} = v_{i,j} + v_{j,i} \tag{56}$$

Introducing equations (56), (55), and (53) into (54), simplifying, and making the argument that the result must hold for arbitrary nodal velocities $v_{Ni}^{(e)}$, we obtain

$$m_{NR}^{(e)}\dot{v}_{Rj}^{(e)} + n_{RMN}^{(e)j}v_{Mj}^{(e)}v_{Ri}^{(e)} + \int_{v_{(e)}} t_{ij}\Psi_{N,j}\,dv = p_{Ni}^{(e)} \tag{57}$$

where

$$n_{RMN}^{(e)j} = \int_{v_{(e)}} \rho\Psi_{R,j}(\mathbf{x})\Psi_M(\mathbf{x})\Psi_N(\mathbf{x})\,dv \tag{58}$$

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* Application of the finite element method to potential flow problems has been considered by Zienkiewicz and Cheung[1], Martin[7], Tong[8], and Oden and Somogyi[9]. Palmer and Asher[10], Tong[8], and Guyan, Ujhara, and Welch[11], considered the problem of interaction of the irrotational motion of an incompressible fluid with an elastic container.

and $m_{NR}^{(e)}$ and $p_{Ni}^{(e)}$ are of the same form as (51) except that F_i and S_i now are interpreted spatially.

Equation (57) represents the equations of motion for a finite element of a fluid media. Because of the convective term $n_{RMN}^{(e)j}v_{Mj}^{(e)}v_{Ri}^{(e)}$ these equations are non-linear in the nodal velocities. The terms $m_{NR}^{(e)}\dot{v}_{Rj}^{(e)} + n_{RMN}^{(e)j}v_{Mj}^{(e)}v_{Ri}^{(e)}$ are dropped in the case of steady flow. The assembly of elements into a single discrete model follows the usual procedure and need not be discussed.

Linear stokesian fluid

As an example of the form equation (57) assumes for a specific type of fluid, we cite the linear Stokesian fluid for which the constitutive equation is given by

$$t_{ij} = \delta_{ij}(-p + \lambda d_{rr}) + 2\mu d_{ij} \tag{59}$$

where p is the hydrostatic pressure and λ and μ are the dilatational and shear viscosities. Introducing (56), (53), and (59) into (57), we have for the equations of motion of the finite element

$$m_{NR}^{(e)}\dot{v}_{Rj}^{(e)} + n_{RMN}^{(e)j}v_{Mj}^{(e)}v_{Ri}^{(e)} + \int_{v_{(e)}} \Psi_{N,i}(\mathbf{x})(-p + \lambda\Psi_{M,r}(\mathbf{x})v_{Mr})\,dv$$
$$+ \int_{v_{(e)}} \mu\Psi_{N,j}(\Psi_{R,i}v_{Rj} + \Psi_{R,j}v_{Ri})\,dv = p_{Ni} \tag{60}$$

Equation (60) represents finite element analogue of the Navier–Stokes equations.

Viscoelastic fluids

As a final example in fluid dynamics, we cite the case of a linear incompressible viscoelastic fluid for which the stress is given by the functional constitutive equation

$$t_{ij} = -p\delta_{ij} + \int_{-\infty}^{t} [\lambda(t-s)\,d_{rr}(s)\delta_{ij} + 2\mu(t-s)\,d_{ij}(s)]\,ds \tag{61}$$

where $\lambda(t-s)$ and $\mu(t-s)$ are material kernals. Then, instead of equation (60) we have

$$m_{NR}^{(e)}\dot{v}_{Rj}^{(e)} + n_{RMN}^{(e)j}v_{Mj}^{(e)}v_{Ri}^{(e)} + \int_{v_e} \Psi_{N,i}\Big\{-p\delta_{ij} + \int_{-\infty}^{t} [\lambda(t-s)\delta_{ij}\Psi_{N,r}v_{Nr}(s)$$
$$+ \mu(t-s)(\Psi_{R,i}v_{Rj}(s) + \Psi_{R,j}v_{Ri}(s))]\,ds\Big\}\,dv = p_{Ni} \tag{62}$$

Thus, the model leads to systems of non-linear integrodifferential equations in the nodal velocities.

REFERENCES

1. O. C. Zienkiewicz and Y. K. Cheung, 'Finite elements in the solution of field problems', *Engineer, Lond.* **222**, 507–510 (1965).
2. R. Courant, K. O. Friedrichs and H. Lewy, 'Über die partiellen Differenzengleichungen der mathematischen Physik,' *Ann. Math.,* **100**, 32 (1928).
3. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics Part I*, McGraw-Hill, New York, 1953, pp. 314–318.
4. K. Haung, *Statistical Mechanics*, John Wiley, New York, 1963, pp. 55–67.
5. D. Greenspan, 'On approximating extremals of functionals. II. Theory and generalizations related to Boundary value problems for non-linear differential equations,' *Int. J. Engng. Sci.,* **5**, 571–588 (1967).
6. A. C. Eringen, *Non-linear Theory of Continuous Media*, McGraw-Hill, New York, 1962, p. 113.

7. H. C. Martin, 'Finite element analysis of fluid flows, *Proc. 2nd Conf. on Matrix Methods in Structural Mechanics*, A.F.I.T. Wright-Patterson Air Force Base (1968).

8. P. Tong 'Liquid sloshing in an elastic container', *A.F.O.S.R.* 66–0943, June 1966.

9. J. T. Oden and D. Somogyi, 'On the application of the finite element method to a class of problems in fluid dynamics,' *J. Engng Mech. Div. ASCE*, 95, EM3 (1969).

10. J. H. Palmer and G. W. Asher, 'Calculation of axisymmetric longitudinal modes for fluid-elastic tank-ullage gas systems and comparison with modal results', *Proc. AIAA. Symp. on Structural Dynamics and Aeroelasticity*, Boston, 1964.

11. R. J. Guran, B. H. Ujihara and P. W. Welch, 'Hydroelastic analysis of axisymmetric systems by a finite element method. *Proc. 2nd Conf. on Matrix Methods in Structural Mechanics*, A.F.I.T., Wright-Patterson Air Force Base (1968).