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ON THE CALCULATION OF CONSISTENT STRESS DISTRIBUTIONS IN FINITE ELEMENT APPROXIMATIONS

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SUMMARY

The theory of conjugate approximations[1] is used to obtain consistent approximations of stress fields in finite element approximations based on displacement assumptions. These consistent stresses are continuous across interelement boundaries and involve less mean error than those computed by the conventional approach.

INTRODUCTION

The problem of calculating stresses in finite element models based on displacement approximations has long been a source of difficulty. Typically, the procedure is as follows.[2]

1. The displacement components u, v and w over a typical finite element are represented approximately as a linear combination of their values (and possibly values of their derivatives) at a prescribed number of nodal points.
2. Following standard procedures,[2] element stiffness matrices $\mathbf{k}^{(e)}$ are calculated using, say, the principle of minimum potential energy.
3. A global stiffness matrix $\mathbf{K}$ is obtained by connecting all of the elements together and applying appropriate boundary conditions. This leads to equilibrium equations governing the connected assembly of elements.
4. The equilibrium equations are solved and stresses in a typical element are evaluated by direct substitution of local displacement field into an appropriate constitutive equation, for example

$$\boldsymbol{\sigma}^{(e)} = \mathbf{D}^{(e)}\,\mathbf{B}^{(e)}\,\boldsymbol{\Omega}^{(e)}\,\mathbf{K}^{-1}\,\mathbf{P} \tag{1}$$

Here $\boldsymbol{\sigma}^{(e)}$ is a vector of local stresss components, $\mathbf{D}^{(e)}$ is the elasticity matrix, $\mathbf{B}^{(e)}$ is an operator appearing in the strain displacement relations, $\boldsymbol{\Omega}^{(e)}$ is a matrix transforming the global generalized displacements $\mathbf{U}$ to the local nodal displacements $\mathbf{u}^{(e)}$ and $\mathbf{P}$ is a vector of generalized forces.

As pointed out by Melosh,[3] stresses computed using the above procedure represent, at best, only averages of the true stresses over a finite element. The stress field so computed will, in general, exhibit finite discontinuities at the boundaries of each element. To evaluate stresses at a nodal point, it is common practice to compute some kind of weighted average of stresses over elements meeting at that node. Wilson[4] noted that such element stresses generally do not represent the true state of stress in any one point of the element: he developed a 'weighted average method' which gave good results for both interior and boundary elements. The paper by Turner and coworkers[5] includes a detailed discussion of an averaging technique, based on an equivalence of nodal forces and element stresses, that overcomes large stress discontinuities in finite elements. Under reasonable smoothness conditions on the local displacement fields, it can be shown that element stresses computed using the procedure outlined previously do indeed converge in the mean to the

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exact values.[6-9] In his 1964 monograph, Gallagher[10] proposed a 'direct' method of formulation of stiffness matrices which involved independent developments related to both stress and displacement fields for an element, and which thereby overcame many of the usual difficulties in interpreting boundary stresses.

In the present paper, we give a consistent procedure for calculating stresses in finite elements which is based on the idea of conjugate approximations.[1] While we make no claim that the method is computationally more efficient than those generally used to compute stresses, it is nevertheless *consistent*, since the distribution of such stresses in the finite element model is directly dictated by the displacement approximations used in formulating element properties. In other words, the stresses so computed are consistent with the displacement approximations. Moreover if the displacement approximations are continuous, the consistent stresses will also be continuous and will be more accurate (in a certain mean square sense) than those computed following conventional procedures.

The procedure for computing consistent stresses arises from the notions of biorthogonal bases for finite-dimensional conjugate spaces. If the function $u(x, y, z)$ (say, a component of displacement) is approximated by functions of the form

$$\tilde{u} = \boldsymbol{\phi}(x, y, z)\,\boldsymbol{\delta} \tag{2}$$

where $\boldsymbol{\phi}(x, y, z)$ is a matrix of suitably chosen interpolation functions and $\boldsymbol{\delta}$ a vector of generalized displacements, then the identification of $\boldsymbol{\phi}(x, y, z)$ defines a basis for a finite-dimensional subspace of the space containing the actual displacement function u. Since the functions $\boldsymbol{\phi}(x, y, z)$ are not orthogonal with respect to an appropriately defined inner product on the space, it is possible to construct a *different set* of so-called conjugate approximation functions $\boldsymbol{\lambda}(x, y, z)$ which form the basis of a space conjugate to that spanned by $\boldsymbol{\phi}(x, y, z)$. The significance of this observation, which is to be used in this paper, is that element stresses should be referred to the conjugate basis rather than mapped completely out of the subspace of the functions $\boldsymbol{\phi}(x, y, z)$, as is done conventionally by the procedure outlined above.

THE CONCEPT OF CONJUGATE APPROXIMATIONS

Although it is not essential that the details of the theory of conjugate approximations be digested in order to apply it to practical computations, we outline briefly here certain of its underlying concepts. Further details can be found in Reference 1.

The displacement field $\mathbf{u}$ described in the previous section can be regarded as an element in an infinite-dimensional linear space $\mathscr{H}$ in which an inner product between two elements $\mathbf{u}$ and $\mathbf{w}$ is defined by

$$(\mathbf{u}, \mathbf{w}) = \int_{\mathscr{R}} \mathbf{u}^{\mathrm{T}}\,\mathbf{w}\,\mathrm{d}\mathscr{R} \tag{3}$$

Here $\mathscr{R}$ is the bounded domain of the functions $\mathbf{u}(x, y, z)$, $\mathbf{w}(x, y, z)$. In most methods of approximation, we deal with a finite-dimensional subspace Φ, instead of $\mathscr{H}$, wherein functions are approximated as linear combinations of a finite number of linearly independent base functions. For example, each displacement component u, v and w can be represented as the sum of G linearly independent functions $\phi_i(x, y, z)$:

$$\left.\begin{aligned} u &\approx \phi_1(x, y, z)\,U_1 + \phi_2(x, y, z)\,U_2 + \ \dots\ + \phi_G(x, y, z)\,U_G \\ v &\approx \phi_1(x, y, z)\,V_1 + \phi_2(x, y, z)\,V_2 + \ \dots\ + \phi_G(x, y, z)\,V_G \\ w &\approx \phi_1(x, y, z)\,W_1 + \phi_2(x, y, z)\,W_2 + \ \dots\ + \phi_G(x, y, z)\,W_G \end{aligned}\right\} \tag{4}$$

where U_i, V_i and W_i are the values of u, v and w (and/or their derivatives) at various nodal points in $\mathscr{R}$. It is often convenient to recast equation (4) in matrix form, as indicated in equation (2), by introducing

$$\left.\begin{aligned} \mathbf{u}^{\mathrm{T}} &= [u, v, w]\,\delta\delta^{\mathrm{T}} = [U_1, U_2, \ldots, U_G, V_1, \ldots, V_G, W_1, \ldots, W_G] \\ \boldsymbol{\Phi} &= \left[\begin{array}{c|c|c} \boldsymbol{\phi} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\phi} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Phi} \end{array}\right], \quad \boldsymbol{\phi} = [\phi_1, \phi_2, \ldots, \phi_G] \end{aligned}\right\} \tag{5}$$

Then

$$\mathbf{u} = \boldsymbol{\Phi}\boldsymbol{\delta} \tag{6}$$

Now corresponding to any finite set of base functions $\phi_i(x, y, z)$, we can always construct a set of conjugate approximation functions which form a countable biorthogonal basis to Φ and its conjugate space Φ^*; that is, a unique set of functions $\boldsymbol{\Lambda}(x, y, z) = [\Lambda_i(x, y, z)]$ which satisfy the biorthogonality condition

$$\int_{\mathscr{R}} \boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{\Lambda}\, \mathrm{d}\mathscr{R} = \mathbf{I} \tag{7}$$

where $\mathbf{I}$ is the $G \times G$ identity matrix. To construct the conjugate functions, we first compute the *fundamental matrix*

$$\mathbf{C} = (\boldsymbol{\phi}, \boldsymbol{\phi}) = \int_{\mathscr{R}} \boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{\phi}\, \mathrm{d}\mathscr{R} \tag{8}$$

Since the elements of $\boldsymbol{\phi}$ are linearly independent, it can be shown that $\mathbf{C}$ is a symmetric, non-singular, positive-definite matrix. Define

$$\boldsymbol{\Lambda}^{\mathrm{T}} = \mathbf{C}^{-1} \boldsymbol{\phi}^{\mathrm{T}} \tag{9}$$

Then

$$\int_{\mathscr{R}} \boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{\Lambda}\, \mathrm{d}\mathscr{R} = \int_{\mathscr{R}} \boldsymbol{\phi}^{\mathrm{T}} \boldsymbol{\phi}\, \mathrm{d}\mathscr{R}\, \mathbf{C}^{-1} = \mathbf{C}\mathbf{C}^{-1} = \mathbf{I} \tag{10}$$

as required. Moreover,

$$\mathbf{C}^{-1} = \int_{\mathscr{R}} \boldsymbol{\Lambda}^{\mathrm{T}} \boldsymbol{\Lambda}\, \mathrm{d}\mathscr{R} \tag{11}$$

The conjugate approximation functions $\boldsymbol{\Lambda}(x, y, z)$ provide a basis for the conjugate subspace Φ^*. Thus elements in Φ^* (i.e. elements appearing as linear functionals of elements in Φ) are linear combinations of the functions $\boldsymbol{\Lambda}(x, y, z)$.

We recall that in finite element approximations, the functions $\boldsymbol{\phi}(x, y, z)$ are sums of elemental approximation functions $\boldsymbol{\psi}^{(e)}(x, y, z)$ that are defined only locally. Thus, for a collection of E elements,

$$\boldsymbol{\phi}(x, y, z) = \sum_{e}^{E} \boldsymbol{\psi}^{(e)}(x, y, z)\, \boldsymbol{\Omega}^{(e)} \tag{12}$$

wherein $\boldsymbol{\Omega}^{(e)}$ is the Boolean transformation appearing in equation (1). However, the functions are generally designed so that $\boldsymbol{\phi}(x, y, z)$ is continuous over $\mathscr{R}$. The local character of the functions $\boldsymbol{\psi}^{(e)}(x, y, z)$ is described by saying that they have *almost disjoint support*. Formally, we would like also to define local conjugate approximation functions $\boldsymbol{\lambda}^{(e)}(x, y, z)$ for the purpose of calculating local approximations of the stress fields over an element. Since, in finite element approximations,

$$\boldsymbol{\Lambda}^{\mathrm{T}} = \mathbf{C}^{-1} \boldsymbol{\phi}^{\mathrm{T}} = \mathbf{C}^{-1} \sum_{e} \boldsymbol{\Omega}^{(e)\mathrm{T}} \boldsymbol{\psi}^{(e)\mathrm{T}} \tag{13}$$

we can set

$$\boldsymbol{\lambda}^{(e)} = \boldsymbol{\Lambda}\boldsymbol{\Omega}^{(e)\mathrm{T}} \tag{14}$$

so that locally

$$\boldsymbol{\lambda}^{(e)} = \mathbf{C}^{-1} \sum_{f} \boldsymbol{\Omega}^{(f)\mathrm{T}} \boldsymbol{\psi}^{(f)\mathrm{T}} \boldsymbol{\Omega}^{(e)\mathrm{T}} \tag{15}$$

Two important observations immediately become available:

1. Each local conjugate function $\boldsymbol{\lambda}^{(e)}(x, y, z)$ is a linear combination of *all* of the local approximations $\boldsymbol{\psi}^{(e)}(x, y, z)$; thus, the functions $\boldsymbol{\lambda}^{(e)}$ do not have disjoint support and are defined over the whole connected domain $\mathscr{R}$.

2. Each local conjugate function is a linear combination of the global functions $\boldsymbol{\phi}(x, y, z)$: thus, they are continuous across inter-element boundaries.

These two properties are markedly different from those of the stresses calculated by means of equation (1).

As a concluding comment in this section, we remark that because of equation (9) the subspaces Φ and Φ^* coincide. Thus, any function u belonging to Φ can be expressed as a linear combination of either the $\boldsymbol{\phi}(x, y, z)$ or the $\boldsymbol{\Lambda}(x, y, z)$. For example,

$$u(x, y, z) = \boldsymbol{\phi}(x, y, z)\,\mathbf{U} = \boldsymbol{\Lambda}(x, y, z)\,\hat{\mathbf{U}} \tag{16}$$

where

$$\mathbf{U} = \int_{\mathscr{R}} \boldsymbol{\phi}^{\mathrm{T}} u \,\mathrm{d}\mathscr{R}, \quad \hat{\mathbf{U}} = \int_{\mathscr{R}} \boldsymbol{\Lambda}^{\mathrm{T}} u \,\mathrm{d}\mathscr{R} \tag{17}$$

and

$$\hat{\mathbf{U}} = \mathbf{C}\mathbf{U} \tag{18}$$

Technically, $\boldsymbol{\Lambda}(x, y, z)\,\hat{\mathbf{U}}$ belongs to the conjugate space Φ^*.

STRESS CALCULATIONS

The strain energy in an elastic finite element is

$$W^{(e)} = \frac{1}{2}\int_{\mathscr{R}} \boldsymbol{\sigma}^{(e)\mathrm{T}} \boldsymbol{\varepsilon}^{(e)} \,\mathrm{d}\mathscr{R} \tag{19}$$

where $\boldsymbol{\sigma}^{(e)}$ is the stress vector and $\boldsymbol{\varepsilon}^{(e)}$ is the vector of strain components. If $\boldsymbol{\Delta}$ is the vector of generalized displacements for the connected model, we have, in accordance with Reference 2,

$$\boldsymbol{\varepsilon}^{(e)} = \mathbf{B}^{(e)}\,\mathbf{u}^{(e)} = \mathbf{B}^{(e)}\,\mathbf{N}^{(e)}\,\mathbf{A}^{(e)}\,\boldsymbol{\Delta} \tag{20}$$

Assuming that elements of $\boldsymbol{\varepsilon}^{(e)}$ belong to the subspace Φ, equation (19) defines a linear functional on Φ and shows that the elements of $\boldsymbol{\sigma}^{(e)}$ belong to Φ^*. Consequently, the functions $\boldsymbol{\sigma}$ should be represented as linear approximations of the conjugate approximation functions $\boldsymbol{\lambda}^{(e)}(x, y, z)$. The validity of the assumption that $\boldsymbol{\varepsilon}^{(e)}$ belongs to Φ rests on the commutativity of the projection Π of $\mathscr{H}$ into Φ and partial differentiation which, for finite element models, is at least approximately true.[1] It follows that a given stress component, say $\sigma_x^{(e)}$, should be represented by

$$\sigma_x^{(e)} = \boldsymbol{\lambda}^{(e)}(x, y, z)\,\bar{\mathbf{S}}^{(e)} \tag{21}$$

where

$$\bar{\mathbf{S}}^{(e)} = \int_{\mathscr{R}} \boldsymbol{\psi}^{(e)\mathrm{T}} \bar{\sigma}_x \,\mathrm{d}\mathscr{R} = (\boldsymbol{\psi}^{(e)\mathrm{T}}, \bar{\sigma}_x) \tag{22}$$

Here $\bar{\sigma}_x$ is the conventional stress component computed from the constitutive equation for the material (i.e. $\bar{\boldsymbol{\sigma}}^{(e)} = \mathbf{D}^{(e)}\boldsymbol{\varepsilon}^{(e)}$).

Procedure

We are now in a position to construct a systematic procedure for computing consistent, continuous stress distributions in finite element approximations. Continuing with the 'conventional' procedure described previously, suppose that the components of displacement over the finite element model are given by equations of the form in equation (6) and that the discontinuous stress field $\bar{\boldsymbol{\sigma}}$ is given by the constitutive law

$$\bar{\boldsymbol{\sigma}} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{DBu} = \mathbf{DB}\boldsymbol{\Phi}\boldsymbol{\Delta} \tag{23}$$

We proceed as follows.

1. If there were unknown displacements at G nodes, construct the $G \times G$ fundamental matrix

$$\mathbf{C} = \int_{\mathscr{R}} \boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi}\,\mathrm{d}\mathscr{R} \tag{24}$$

where $\boldsymbol{\phi}$ is the row matrix of interpolation functions. If the body is homogeneous and of mass density ρ, the matrix $\mathbf{C}$ may be already available since it is $1/\rho$ times a submatrix on the diagonal of the consistent mass matrix for the finite element approximation of the body.

2. Construct the $1 \times G$ matrices of conjugate approximation functions

$$\boldsymbol{\Lambda} = \boldsymbol{\phi}\mathbf{C}^{-1} \tag{25}$$

3. The vector $\bar{\boldsymbol{\sigma}}$ of discontinuous (conventional) stress components is of order $R \times 1$, where $R = 6$ for a three-dimensional solid, $R = 3$ for plane stress, $R = 1$ for a straight rod, etc. Construct the $R \times G$ matrix $\mathbf{R}$ defined by

$$\mathbf{R} = \int_{\mathscr{R}} \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\phi}\,\mathrm{d}\mathscr{R} \tag{26}$$

The matrix $\mathbf{R}$ has G columns, one corresponding to each node.

4. The consistent stress distribution over the entire collection of elements is now computed by the formula

$$\boldsymbol{\sigma} = \mathbf{R}\boldsymbol{\Lambda}^{\mathrm{T}} \tag{27}$$

However, in practice it is sufficient to have only the consistent stresses at each node. These are obtained automatically by computing

$$\mathbf{S} = \mathbf{R}\mathbf{C}^{-1} \tag{28}$$

The columns of the $R \times G$ matrix $\mathbf{S}$ contain the values of consistent stress at each of the G nodal points.

EXAMPLES

To demonstrate the basic ideas, we now examine applications of the theory to selected problems. Consider first the one-dimensional example of a straight bar with a linearly modulus of elasticity, $E(x) = E_0(1+x)$ subjected to a prescribed longitudinal displacement of $u(x) = \alpha(1 - x^2/36)$. The exact stress distribution is $\sigma(x) = E(x)\,\mathrm{d}u/\mathrm{d}x = -\alpha E_0 x(1+x)/18$. To analyse the bar using finite elements, we adopt a rather crude displacement approximation consisting of six elements over each of which the displacement is assumed to vary linearly; this problem then amounts to a slight generalization of one described in Reference 1. In this case, the approximate displacement field is given by

$$u = \boldsymbol{\phi}(x)\,\mathrm{U} \tag{29}$$

where

$$\left.\begin{aligned} &\phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_7(x)] \\ &\phi_1(x) = 1-x, \quad 0 \leqslant x \leqslant 1 \\ &\phi_i(x) = x+2-i+2(i-x)\,h(x-i+1), \quad i-1 \leqslant x \leqslant i+1 \\ &\phi_7(x) = 6+x, \quad 5 \leqslant x \leqslant 6 \end{aligned}\right\} \tag{30}$$

$h(x-i)$ being the unit step function, $\phi_k(x) = 0$ elsewhere, and

$$U = \frac{\alpha}{36}\{36.35, 32.27, 20.11, 0\} \tag{31}$$

Since the conventional, discontinuous stresses $\bar{\sigma}(x)$ are simply

$$\bar{\sigma}(x) = E_0(1+x)\frac{\mathrm{d}\boldsymbol{\phi}(x)}{\mathrm{d}x}\mathbf{U} \tag{32}$$

we can now generate the matrix $\mathbf{S}$ of consistent nodal stresses by introducing equations (29)–(31) into equation (26):

$$\mathbf{R} = \int_0^6 \bar{\sigma}(x)\,\boldsymbol{\phi}(x)\,\mathrm{d}x = -\frac{E_0\alpha}{216}[5, 26.74, 146, 242, 362, 220] \tag{33}$$

We next calculate the fundamental matrix $\mathbf{C}$ of equation (24) and then its inverse

$$\mathbf{C}^{-1} = \left[\int_0^6 \boldsymbol{\phi}^{\mathrm{T}}\boldsymbol{\phi}\,\mathrm{d}x\right]^{-1} = \frac{1}{390}\begin{bmatrix} 1351 & -362 & 97 & -26 & 7 & -2 & 1 \\ & 724 & -194 & 52 & -14 & 4 & -2 \\ & & 675 & -182 & 45 & -14 & 7 \\ & & & 676 & -182 & 52 & -26 \\ & & & & 679 & -194 & 97 \\ & & & & & 724 & -362 \\ & \text{Sym.} & & & & & 1351 \end{bmatrix} \tag{34}$$

Then, the consistent stresses at each node are

$$\mathbf{S} = \mathbf{R}\mathbf{C}^{-1} = -\frac{E_0\alpha}{36}[0{\cdot}8, 3, 11{\cdot}1, 23{\cdot}8, 39{\cdot}2, 60{\cdot}9, 80{\cdot}0] \tag{35}$$

Finally, the complete stress distribution $\sigma(x)$ is $\mathbf{S}\boldsymbol{\phi}^{\mathrm{T}}$.

The results of these calculations are shown in Figure 1. It is seen that the consistent stresses are continuous and lead to a better approximation of the maximum stress.

As a final example, we demonstrate that once the conventional element stresses are known, improved continuous stress distributions can be obtained by using almost any convenient set of conjugate approximation functions $\boldsymbol{\phi}$ and $\boldsymbol{\Lambda}$ in the construction of equations (26) and (27). The matrix $\boldsymbol{\phi}$ used in such calculations need not correspond to that used to obtain the conventional stresses, even though the actual consistent stresses computed as described previously will lead to the most accurate approximation for a given $\boldsymbol{\phi}(x, y, z)$. Computationally, however, the use of lower-order approximations in $\boldsymbol{\phi}$ to compute stresses may be more practical and will generally provide better stresses than those given by conventional methods. Mathematically, such approximations amount to projecting the conventional stresses into a subspace different than that spanned by the original collection of approximating functions. The results of such calculations applied

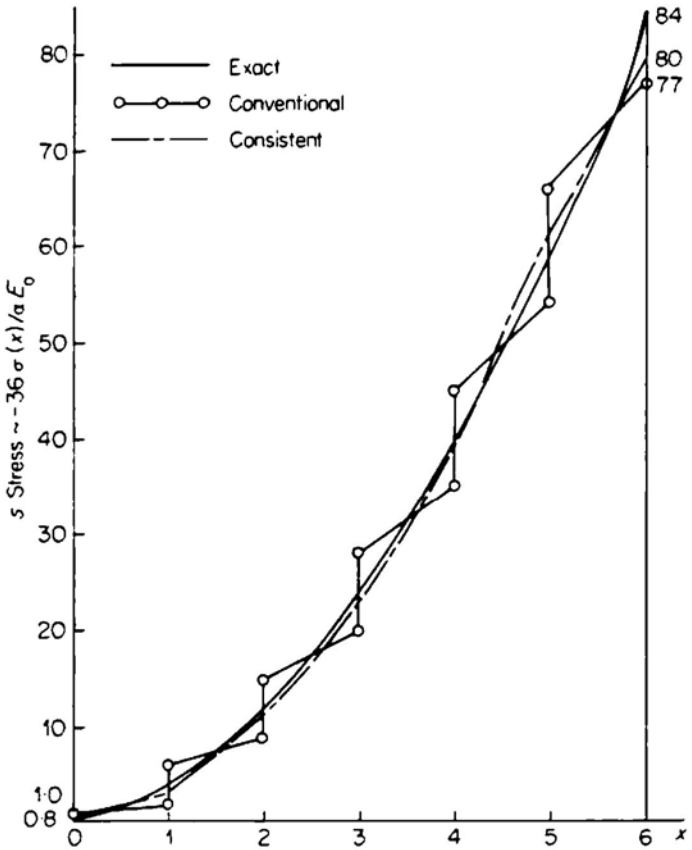

Figure 1. Comparison of conventional finite element stresses with consistent stresses

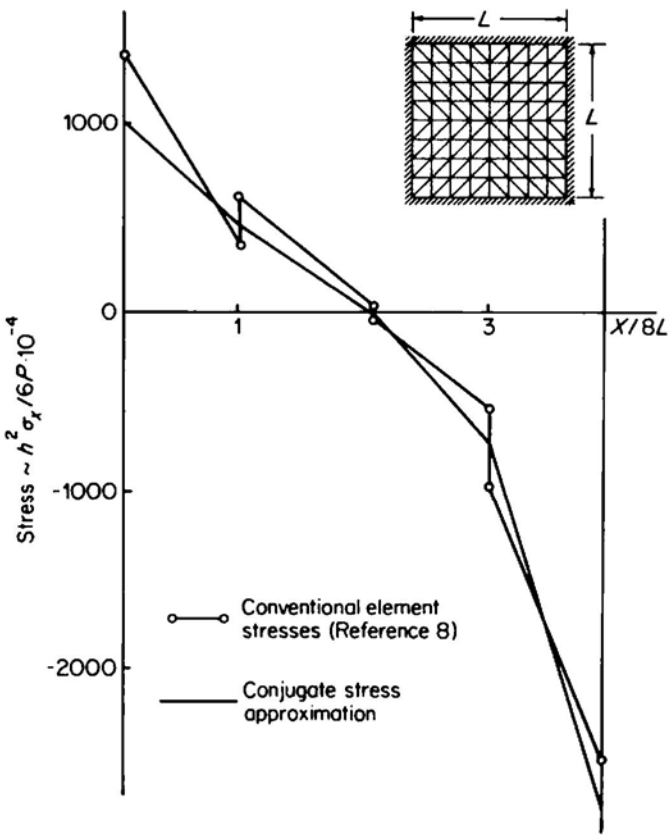

Figure 2. Normal stresses at top fibres along centre line in a centrally loaded square plate

to the problem of bending of a clamped, square plate subjected to a centrally located concentrated force P are shown in Figure 2. Here the conventional stresses are those obtained by Bazeley and co-workers[11] using non-conforming elements. The improved continuous stress distribution shown was obtained by using the very crude but conforming set of piecewise linear interpolation functions in equations (26) and (27). It is observed that even in this case the conjugate stress approximations tend to give better estimates of stresses near points of stress concentration.

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APPENDIX

The proof that the best approximation in a mean square sense arises from the use of conjugate approximations was presented in Reference 1 in connection with quadratic functionals defined on self-conjugate spaces. Here we generalize this result to the case in which a linear space and its dual are considered. We first consider cases in which the character of the spaces involved is arbitrary, and then we apply the result to linear elasticity.

Let $\mathscr{F}$ and $\mathscr{G}$ denote dual linear spaces the elements of which are functions denoted f and g respectively. Let (f, g) denote the inner product on $\mathscr{F}$ and $\mathscr{G}$. Further, let Φ denote a finite-dimensional subspace of $\mathscr{F}$, and Ψ its dual in $\mathscr{G}$. The spaces Φ and Ψ are spanned by biorthogonal bases $\phi_k \in \Phi$, $\psi^k \in \Psi$ such that

$$(\phi_k, \psi^m) = \delta_k^m, \quad m, k = 1, 2, \ldots, n \tag{36}$$

Then, if F and G are the projections of f and g into Φ and Ψ,

$$F = F^k \phi_k, \quad G = G_m \psi^m \tag{37}$$

wherein the repeated indices are summed from 1 to n and

$$F^k = (f, \psi^k), \quad G_m = (g, \phi_m) \tag{38}$$

Let $\Lambda^k \phi_k$ and $M_m \psi^m$ denote arbitrary elements in Φ and Ψ and consider the bilinear functional

$$J[\Lambda^k, M_m] = (f - \Lambda^k \phi_k, g - M_m \psi^m) \tag{39}$$

which, upon simplification, can be written

$$J[\Lambda^k, M_m] = (f - F, g - G) + (F^k - \Lambda^k)(G_k - M_k) \tag{40}$$

Obviously, either of the projections defined by equation (36) makes the second term in equation (40) vanish. If we now introduce a linear, positive-definite, regular mapping κ from $\mathscr{F}$ to $\mathscr{G}$, such that $\kappa(\Phi) \subset \Psi$, then κ defines a matrix $\kappa_{mk} = \{\phi_m, \kappa(\phi_k)\}$ such that

$$\kappa_{mk} \Lambda^k = M_m \tag{41}$$

Then $J[\Lambda^k, M_m] = J[\Lambda^k, k_{mk} \Lambda^k]$ becomes a positive-definite quadratic form

$$J[\Lambda^k] = \{f - F, k(f - F)\} + k_{mk}(\Lambda^m - F^m)(\Lambda^k - F^k) \tag{42}$$

Consequently equation (42) assumes a minimum value when Λ^k is chosen according to equation (38).

In the case of linear elasticity, let f correspond to the stress tensor σ_{ij} and g correspond to the strain tensor ε_{ij}. Then (f, g) represents, with a constant, the strain energy. Let ε_{ij} be the strain

resulting from a finite element model of the displacement field. Then, in view of the above results, the best approximation of σ_{ij}, in the sense of minimizing the square error equation (42), is that function in Φ for which the coefficients are computed using equation (38).

REFERENCES

1. H. J. Brauchli and J. T. Oden, 'Conjugate approximation functions in finite element analysis', *Q. appl. Math.* to appear.
2. O. C. Zienkiewicz and Y. K. Chung, *The Finite Element Method in Structural and Continuum Mechanics*, McGraw–Hill, London, 1967.
3. R. J. Melosh, 'Basis of derivation of matrices for the direct stiffness method', *AIAA Jnl.* 1, 1631–1637 (1963).
4. E. L. Wilson, *Finite Element Analysis of Two-dimensional Structures*. Ph.D. Thesis, University of California, Berkeley, 1963.
5. M. J. Turner, H. C. Martin and B. C. Weikel, 'Further developments and applications of the stiffness method', *Matrix Meth. Struct. Analysis*, AGARD 72, 203–266 (1964).
6. M. W. Johnson, Jr. and R. W. McLay, 'Convergence of the finite element method in the theory of elasticity', *J. appl. Mech.* 35, E. 2, 274–278 (1968).
7. S. W. Key, *A Convergence Investigation of the Direct Stiffness Method*, Ph.D. Thesis, University of Washington, Seattle, 1966.
8. E. R. A. Oliviera, 'Theoretical foundations of the finite element method', *Int. J. Solids Struct.* 4, 929–952 (1968).
9. P. Tong and T. H. H. Pian, 'The convergence of the finite element method in solving linear elasticity problems', *Int. J. Solids Struct.* 3, 865–879 (1967).
10. R. H. Gallagher, *A Correlation Study of Methods of Matrix Structural Analysis*, Pergamon Press, London, 1964.
11. G. P. Bazeley, Y. K. Cheung, B. M. Irons and O. C. Zienkiewicz, 'Triangular elements in plate bending-conforming and non-conforming solutions', *Proc. Matrix Meth. Struct. Mech.* TR-66-80, 547–576 (1966).