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PENALTY-FINITE ELEMENT METHODS FOR CONSTRAINED PROBLEMS IN ELASTICITY

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Preface

I began studying exterior penalty methods as a basis for finite element methods around three years ago with the able help of my colleague and former student, Professor Noboru Kikuchi, now at the University of Michigan. At the kind invitation of Professor Kardestuncer, I delivered a preliminary report of our work at the Symposium on the Unification of Finite Elements, Finite Differences, and the Calculus of Variations held at the University of Connecticut in May, 1980. Later, Kikuchi and I expanded that report and rewrote it in a style we hoped would be accessible to a broad audience of engineers who may wish to consider these and various related methods for the numerical analysis of problems in fluid and solid mechanics. This work is to be published in full in the International Journal for Numerical Methods in Engineering. The present document, which has been written to provide participants of the Symposium on Finite Elements at Hefei, The People's Republic of China, with a detailed account of my lectures given at this meeting, is taken more or less verbatim from the joint IJNME paper written with Kikuchi, only minor editorial changes being made in a few places.

I would like to record a special note of thanks to the Chinese Mechanical Engineering Society, the Chinese Mechanics Society, the Chinese Mathematics Society, Professor H. Kardestuncer for their gracious invitation to participate in the Symposium and to lecture on my work.

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CONTENTS

INTRODUCTION

In this paper, we shall consider several complications that arise in applications of finite element methods to equilibrium problems in elasticity in which various constraints are imposed on the motion. For clarity, special attention will be given to the constraint of incompressibility in linear elasticity, but some discussion of unilateral constraints for contact problems will also be given. Some of the work reported here summarizes recent results of Oden, Kikuchi and Song[14] where a detailed analysis of penalty methods for constrained problems in linear elasticity can be found.

Portions of this paper is expository in nature and deal with properties of approximations of somewhat general problems with constraints. We discuss conditions under which such constrained problems admit unique solutions and conditions for the existence of multipliers in Lagrange-multiplier formulations. These, in turn, lead to criteria for stability and convergence of mixed and penalty methods. We choose equilibrium problems in elasticity as a convenient area of application of these ideas with constraints.

We give special attention to the Reduced Integration-Penalty Methods (RIP) for several reasons: first, they are very popular and, when they work, can be very effective; second, they are not well understood; and, third, they constitute the best example of numerical methods that cannot be completely evaluated, judged, or understood on purely the basis of numerical experiments - an analysis of their stability and convergence properties is important to their successful use.

It is not our aim to deal with mathematical issues of a very deep nature here, since we wish to make the ideas and results accessible to a wide audience. Nevertheless, some of the mathematical notions we use deserve a brief review and we have, on the advise of colleagues who read earlier versions, of this work, listed some of these in an appendix.

2. CONSTRAINED VARIATIONAL PROBLEMS

We begin by considering a classical miminization problem in the calculus of variations. Given

V = a space of "admissible functions" (a real Hilbert space),

$F:V \to \mathbb{R}$ a functional defined on V, and

K = a nonempty closed subset of V,

find $u \in K$ such that, for any v in K, F assumes its minimum value at u:

$$u \in K \ : \ F(u) \leq F(v) \quad \forall\, v \in K \tag{2.1}$$

It is well known that there will exist a unique minimizer of F on K whenever the following four (minimization) conditions hold:

M.1) K is convex; i.e. if u and v belong to K, then $\theta u + (1-\theta)v \in K$, $0 < \theta < 1$.

M.2) F is strictly convex; i.e. for $0 < \theta < 1$ and $u \neq v$,
$F(\theta u + (1-\theta)v) < \theta F(u) + (1-\theta)F(v)$

M.3) F is differentiable on K; i.e. for each $u \in K$ there exists an operator $DF(u):V \to V'$ such that

$$\lim_{\theta \to 0^+} \frac{\partial F(u+\theta v)}{\partial \theta} = \langle DF(u),v\rangle \ , \quad \forall\, v \in V$$

where V' is the dual space of V, and $\langle\cdot,\cdot\rangle$ denotes duality pairing on $V' \times V$ (i.e. $\langle DF(u),v\rangle$ is the "first variation" in F at u in direction v)

M.4) F is coercive; i.e. for $v \in K$,
$F(v) \to +\infty$ as $\|v\|_V \to \infty$
where $\|\cdot\|_V$ is the norm on V.

Virtually all of these conditions can be weakened (with the possible loss of uniqueness); the <u>constraint set</u> K need only be weakly sequentially closed, F need not be convex nor differentiable, but should be lower semicontinuous in some sense. See Oden and Kikuchi[13] for further details.

Now an important aspect of the minization problem (2.1) in the case that M.1 - M.3 hold is that the minimizer u of F can be characterized as the solution of a <u>variational inequality</u>:

$$u \in K: \langle DF(u), v-u \rangle \geq 0 \quad \forall v \in K \tag{2.2}$$

Problems (2.1) and (2.2) are equivalent: any solution of (2.1) satisfies (2.2) and vice-versa (whenever conditions M.1-M.3 hold). In the special case in which K is a linear subspace of V, (2.2) reduces to the <u>variational equality</u>.

$$u \in K: \langle DF(u), v \rangle = 0 \quad \forall v \in K \tag{2.3}$$

from which Euler equations for the variational problem can be derived.

In order to simplify the discussion, we shall, from this point onward, consider a restricted class of problems in which the following conventions and assumptions are in force:

N.1 The functional $F:V \to \mathbb{R}$ is a quadratic functional of the form

$$F(v) = \frac{1}{2} a(v,v) - f(v) \qquad v \in V \tag{2.4}$$

where $a(\cdot,\cdot)$ is a symmetric bilinear form mapping $V \times V$ into $\mathbb{R}$ and f is continuous linear functional on V (i.e. $f \in V'$)

N.2 The bilinear form $a(\cdot,\cdot)$ is continuous and V-elliptic; i.e. there exist positive constants M and m such that

$$\begin{aligned} a(u,v) &\leq M \|u\|_V \|v\|_V \quad && \forall u,v \in V \\ a(v,v) &\geq m \|v\|_V^2 && \forall v \in V \end{aligned} \tag{2.5}$$

We easily verify that if N.1 and N.2 hold, then F is strictly convex and differentiable. Indeed,

$$\langle DF(u),v\rangle = a(u,v) - f(v) \quad \forall\ u,v \in V \tag{2.6}$$

Moreover, since

$$f(v) \leq \|f\|_{V'}\ \|v\|_V \tag{2.7}$$

where $\|\cdot\|_{V'}$ is the norm on the dual space V' of V, we must have

$$F(v) \geq \frac{m}{2}\ \|v\|_V^2 - \|f\|_{V'}\ \|v\|_V$$

for any $v \in V$. Hence, $F(v) \to +\infty$ as $\|v\|_V \to \infty$. It follows that whenever conditions N.1 and N.2 hold, conditions M.2 - M.3 hold. Thus, conditions N.1 and N.2 are sufficient to guarantee the existence of a unique solution to the variational inequality (recall (2.2) and (2.6))

$$u \in K: \quad a(u,v-u) \geq f(v-u) \quad \forall\ v \in K \tag{2.8}$$

where K is a closed convex subset of V. Again, the solution u of (2.8) is the unique minimizer of the functional $F(v) = \frac{1}{2}\,a(v,v) - f(v)$ in K.

Examples: The important minimization problems of interest here are those arising in linear and nonlinear elasticity. Consider a homogeneous, isotropic linearly elastic body Ω with smooth boundary Γ subjected to body forces f_i and surface tractions S_i on a portion Γ_F and fixed along a portion Γ_D of Γ. The strain $\varepsilon_{ij}(\underset{\sim}{v})$ produced by a displacement field $\underset{\sim}{v}$ is, of course, $\varepsilon_{ij}(\underset{\sim}{v}) = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$, $1 \leq i,j \leq 3$, and the stress is

$$\sigma_{ij}(\underset{\sim}{v}) = (\kappa - \frac{2\mu}{3})\delta_{ij}\varepsilon_{kk}(\underset{\sim}{v}) + 2\mu\varepsilon_{ij}(\underset{\sim}{v})$$

$$= \kappa\delta_{ij}\varepsilon_{kk}(\underset{\sim}{v}) + 2\mu\varepsilon^D_{ij}(\underset{\sim}{v})$$

where κ is the bulk modulus of the material, μ is the shear modulus, and $\varepsilon^D_{ij}(\underset{\sim}{v})$ is the derivatoric strain,

$$\varepsilon^D_{ij}(\underset{\sim}{v}) = \varepsilon_{ij}(\underset{\sim}{v}) - \frac{1}{3}\delta_{ij}\varepsilon_{kk}(\underset{\sim}{v}) \quad (\text{for } \Omega \subset \mathbb{R}^3).$$

In this case, we take for the space of admissible displacements,

$$V = \{\underset{\sim}{v} = (v_1, v_2, v_3) \,|\, v_i \in H^1(\Omega) \ , \ v_i = 0 \text{ a.e. on } \Gamma_D \ , \ i = 1,2,3\} \tag{2.9}$$

with norm

$$\|v\|_V = \|v\|_1 \equiv \left\{ \int_\Omega v_{i,j} v_{i,j} \, dx \right\}^{1/2} \tag{2.10}$$

where $dx = dx_1 dx_2 dx_3$, $H^1(\Omega)$ is the standard Sobolev space of functions with square-integrable generalized derivatives, and meas $\Gamma_D > 0$. The functional F is now the total potential energy

$$F(\underset{\sim}{v}) = \frac{1}{2} a(\underset{\sim}{v},\underset{\sim}{v}) - f(\underset{\sim}{v}) \ , \ \underset{\sim}{v} \in V$$

$$a(\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega [2\mu\varepsilon^D_{ij}(\underset{\sim}{u}) \ \varepsilon^D_{ij}(\underset{\sim}{v}) + \kappa\varepsilon_{kk}(\underset{\sim}{u})\varepsilon_{mm}(\underset{\sim}{v})]dx \tag{2.11}$$

Here f is the work done by the external forces. Assuming $f_i \in L^2(\Omega)$, $S_i \in L^2(\Gamma_F)$ (Γ_F being a smooth surface), then

$$f(v) = \int_\Omega f_i v_i \, dx + \int_{\Gamma_F} S_i v_i \, ds \qquad (2.12)$$

As examples of constraints, we mention:

1. Incompressibility. In this case, K is a linear subspace of V ,

$$K = K_1 = \{\underset{\sim}{v} \in V | \operatorname{div} \underset{\sim}{v} = 0 \text{ a.e. in } \Omega\} \qquad (2.13)$$

where $\operatorname{div} \underset{\sim}{v} = \varepsilon_{kk}(\underset{\sim}{v})$ is the divergence of the displacement field $\underset{\sim}{v}$.

The problem then is to

Find $u \in K_1$ such that

$$a(\underset{\sim}{u},\underset{\sim}{v}) = f(\underset{\sim}{v}) \quad \forall \underset{\sim}{v} \in K_1 \qquad (2.14)$$

where, whenever $\underset{\sim}{u},\underset{\sim}{v} \in K_1$

$$a(\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega 2\mu \, \varepsilon^D_{ij}(\underset{\sim}{u})\varepsilon^D_{ij}(\underset{\sim}{v}) \, dx. \qquad (2.15)$$

2. Unilateral Contact. We consider situations in which the body comes in contact with a rigid frictionless foundation a (normalized) distance s away from the body in its initial configuration. If Γ_C is the contact surface, then

$$K = K_2 = \{\underset{\sim}{v} \in V | \underset{\sim}{v} \cdot \underset{\sim}{n} - s \leq 0 \text{ a.e. on } \Gamma_C\} \qquad (2.16)$$

$\underset{\sim}{n}$ being a unit outward normal to Γ_C . In this case, K is a nonempty closed convex subset of V and $\underset{\sim}{v} \cdot \underset{\sim}{n}$ is interpreted in the sense of traces of functions in V onto Γ .

The statement of the variational boundary value problem for contact of a compressible elastic body is

Find $\underset{\sim}{u} \in K_2$ such that

$$a(\underset{\sim}{u},\underset{\sim}{v} - \underset{\sim}{u}) \geq f(\underset{\sim}{v}-\underset{\sim}{u}), \quad \forall \underset{\sim}{v} \in K_1 \tag{2.17}$$

If the material in the contact problem is also incompressible, we have

Find $\underset{\sim}{u} \in K_1 \cap K_2$ such that

$$\int_\Omega 2\mu\varepsilon^D_{ij}(\underset{\sim}{u})\ \varepsilon^D_{ij}(\underset{\sim}{v}-\underset{\sim}{u})dx \geq f(\underset{\sim}{v}-\underset{\sim}{u}) \quad \forall \underset{\sim}{v} \in K_1 \cap K_2 \tag{2.18}$$

3. ALTERNATE VARIATIONAL FORMULATIONS

While algorithms can be devised to solve the special variational problems (2.11), (2.14), and (2.15) directly, there are two disadvantages in these formulations from a practical point of view: 1) the direct approximation of these variational problems requires that we somehow approximate the constraint at sets K_1, K_2, or $K_1 \cap K_2$ and this is generally quite difficult, and 2) these particular formulations do not employ Lagrange multipliers for handling the constraints and in each case the multipliers has a definite physical interpretation. For example, in the case of the incompressibility constraint, the Lagrange multiplier is the hydrostatic pressure, and this pressure must be known in order that the stress be determined. In short, in physical problems with constraints, the multipliers associated with the constraints are often equally as important as the minimizers of the energy functional themselves. For this reason, it is standard practice to consider Lagrange multiplier techniques for such problems.

3.1 Lagrange Multipliers. We consider again the minimization problem (2.1) with F given by (2.4), conditions N.1 and N.2 in force, and K now defined by

$$K = \{v \in V | Bv = g_o\} \tag{3.1}$$

where B is the linear continuous constraint operator, $B:V \to Q$, Q being a Hilbert space, and g_o given data in the range $R(B) \subset Q$ of B. The classical Lagrange multiplier approach to (3.1) consists of seeking saddle points (u,p) of the Lagrangian

$$L:V \times Q' \to \mathbb{R}; \quad L(v,q) = F(v) - [q,Bv-g_o] \tag{3.2}$$

where Q' is the dual of Q and $[\cdot,\cdot]$ denotes duality pairing on $Q' \times Q$. The saddle point (u,p) will satisfy

$$L(u,q) \leq L(u,p) \leq L(v,p) \quad \forall\, q \in Q', \quad \forall\, v \in V \tag{3.3}$$

The major issue at this point is: when will there exist a unique saddle point (u,p) of L such that u satisfies (2.1)? It is not difficult to show (see, e.g. Ekeland and Temam[3]) that sufficient conditions for the existence of a solution to (3.3) are:

S.1) Conditions M.2, M.3, M.4 hold for the functional F for K = V

S.2) There exists a $v_o \in V$ such that

$$L(v_o,q) \to -\infty \quad \text{as} \quad \|q\|_{Q'} \to \infty, \ \forall\, q \in Q'$$

Thus, as we noted earlier, whenever N.1 and N.2 hold, condition S.1 satisfied. Moreover, if S.1) and S.2) hold, the unique saddle point (u,p) is characterized by the equations

$$\langle DF(u),v\rangle - [p,Bv] = 0 \quad \forall\, v \in V$$
$$[q,Bu] = [q,g_o] \quad \forall\, q \in Q' \tag{3.4}$$

or, since F is given by (2.4),

$$a(u,v) - [p,Bv] = f(v) \quad \forall\, v \in V$$
$$[q,Bu] = [q,g_o] \quad \forall\, q \in Q' \tag{3.5}$$

For example, in the case of the incompressibility constraint in linear elasticity, we have

$$a(\underset{\sim}{u},\underset{\sim}{v}) = \int_\Omega 2\mu\, \varepsilon^D_{ij}(\underset{\sim}{u})\, \varepsilon^D_{ij}(\underset{\sim}{v})\, dx$$
$$[q,Bv] = \int_\Omega q \,\mathrm{div}\, \underset{\sim}{v}\, dx \tag{3.6}$$

In this case, V is given by (2.9), f by (2.12), $Q = L^2(\Omega) = Q'$, $[\cdot,\cdot] = (\cdot,\cdot) =$ the L^2- inner product, and $B = \mathrm{div}: V \to Q$ $(g_o=0)$.

<u>3.2 Perturbed Lagrangian</u>. A major problem with the Lagrange multiplier method is that the coerciveness condition S.2) may not generally hold. We are then not guaranteed the existence of a unique Lagrange multiplier p . To overcome this difficulty, one can <u>regularize</u> the problem by introducing a perturbed Lagrangian L : let ε be an arbitrary positive number, and define

$$L_\varepsilon(v,q) = L(v,q) - \tfrac{1}{2}\varepsilon\, \|q\|^2_{Q'} \tag{3.7}$$

Clearly, S.1) and S.2) now hold. Thus for each $\varepsilon > 0$, there exists a unique saddle point $(u_\varepsilon,p_\varepsilon)$ of L_ε characterized by

$$a(u_\varepsilon,v) - [p_\varepsilon,Bv] = f(v) \quad \forall\, v \in V$$
$$\varepsilon[q,j^{-1}(p_\varepsilon)] + [q,Bu_\varepsilon] = [q,g_o] \quad \forall\, q \in Q' \tag{3.8}$$

where j^{-1} is the inverse of the Riesz map $j:Q \to Q'$.

The next issue is to determine if the sequence $\{(u_\varepsilon,p_\varepsilon)\}$ of solutions to (3.8) will converge to the unique saddle point of L ; i.e., to the solution of (3.5).

Since $(u_\varepsilon,p_\varepsilon)$ is a saddle point of L_ε , we must have

$$F(u_\varepsilon) - [q,Bu_\varepsilon] - \frac{\varepsilon}{2}\,\|q\|^2_{Q'}$$
$$\leq F(v) - [p_\varepsilon,Bv] - \frac{\varepsilon}{2}\,\|p_\varepsilon\|^2_{Q'}$$

for any $q \in Q'$ and $v \in V$. Without loss of generality, we assume throughout this study that $\ker B \neq \{0\}$. Then it is possible to choose $v_o \neq 0$ such that $Bv_o = 0$. Choosing also $q = 0$, we have

$$F(u_\varepsilon) \leq F(v_o) - \frac{\varepsilon}{2}\,\|p_\varepsilon\|^2_{Q'} \leq F(v_o) \tag{3.9}$$

But since F is coercive (recall M.4), this bound implies that a constant C , independent of ε , exist such that

$$\|u_\varepsilon\|_V \leq C \quad \text{for all } \varepsilon > 0$$

However (see item ii) in the appendix), this guarantees the existence of a subsequence $\{u_{\varepsilon'}\}$ of solutions and an element $u \in V$ such that $u_{\varepsilon'}$ converges weakly to u as ε' tends to zero:

$$u_{\varepsilon'} \rightharpoonup u$$

Observe that by $(3.8)_2$,

$$[p_\varepsilon, Bu_\varepsilon - g_o] = -\varepsilon[p_\varepsilon, j^{-1}(p_\varepsilon)] = -\varepsilon \|p_\varepsilon\|^2_{Q'} \leq 0$$

Hence, since $L_\varepsilon(u_\varepsilon, p_\varepsilon) \leq L_\varepsilon(v, p_\varepsilon)$ for arbitrary v, we have

$$F(u_\varepsilon) \leq F(v) + [p_\varepsilon, Bu_\varepsilon - g_o] \leq F(v) \quad \forall\, v \in K$$

It follows that the weak limit u of the subsequence $\{u_{\varepsilon'}\}$ is precisely the minimizer of the functional F (since $F(u) \leq \liminf_{\varepsilon' \to 0} F(u_{\varepsilon'}) \leq F(v)$ $\forall\, v \in K$).

3.3 Troubles with the Multipliers-The Babuska-Brezzi Condition. To obtain a unique multiplier p as the weak limit of the subsequence $\{p_{\varepsilon'}\}$, we must also show that $\|p_{\varepsilon'}\|_Q$, is uniformly bounded in ε'. Unfortunately, this is generally impossible, for reasons we shall now explain, and it is necessary to add another condition to our theory to overcome such difficulties. We continue to choose $g_o = 0$.

Let $B^*: Q' \to V'$ denote the transpose of the constraint operator B:

$$\langle B^*q, v\rangle = [B, Bv] \quad \forall\, v \in V,\ \forall\, q \in Q' \tag{3.10}$$

and set

$$\ker B^* = \{q \in Q' |\ \langle B^*q, v\rangle = 0 \quad \forall\, v \in V\} \tag{3.11}$$

If $\ker B^* \neq 0$, we can never expect the solution p to (3.5) to be unique, since the addition to p of any element in $\ker B^*$ would also satisfy these equations. To overcome this difficulty, we introduce the quotient space $Q'/\ker B^*$, the elements of which are equivalence classes (cosets) defined by

$$\dot{p} = \{q \varepsilon Q' \mid p - q \varepsilon \ker B^*\} \qquad (3.12)$$

The space $Q'/\ker B^*$ is a Banach space with norm

$$\|\dot{p}\|_{Q'/\ker B^*} = \inf_{q \varepsilon \ker B^*} \|p + q\|_{Q'} \qquad (3.13)$$

Since $[p,Bv] = [p+q,Bv]$ $\forall$ $q \varepsilon \ker B^*$, we will use the notation $[\dot{p},Bv]$ for representing $[p + q,Bv]$ for every $q \varepsilon \ker B^*$. That is, the multiplier p in (3.5) can only be determined to within an element in $\ker B^*$. However, the equivalence class $\dot{p}$ may be unique in $Q'/\ker B^*$. Thus we write

$$a(u,v) - [\dot{p},Bv] = f(v) \ , \ \forall \ v \ \varepsilon \ V$$

$$[q,Bu] = 0 \ , \ \forall \ q \ \varepsilon \ Q'$$

$$a(u_\varepsilon,v) - [p_\varepsilon,Bv] = f(v), \ \ \forall \ v \ \varepsilon \ V \qquad (3.14)$$

$$[q,\varepsilon j^{-1}(p_\varepsilon) + Bu_\varepsilon] = 0, \ \ \forall \ q \ \varepsilon \ Q'$$

in place of (3.5) and (3.8).

We still must show that the sequences $\{p_\varepsilon\}$ are uniformly bounded in ε in some sense. For this purpose, we shall introduce the so-called Babuska-Brezzi condition, which plays a fundamental role in the theory of elliptic equations*:

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*The importance of conditions of this type in the theory of elliptic equations[1-4] and their approximation was first demonstrated by BABUSKA, who used this circle of ideas in many diverse applications. Further work on this area was done in a well-known and important paper by BREZZI[5]. A similar condition for the special case of Stokes' problem with the incompressibility condition div $\underline{v} = 0$ was studied somewhat earlier by LADYSZHENSKAYA, and the condition in connection with incompressible flows has been referred to as the LBB-(LADYSZHENSKAYA-BABUSKA-BREZZI) condition in that context.

There exists a constant $\alpha>0$ such that

$$\alpha \inf_{q \,\varepsilon\, \ker B^*} \|p + q\|_{Q'} \quad (\equiv \alpha \|\dot{p}\|_{Q'/\ker B^*})$$

$$\leq \sup \frac{[p,Bv]}{\|v\|_V} \quad \forall\, p \,\varepsilon\, Q' \tag{3.15}$$

Clearly, when (3.14) and (3.15) hold,

$$\alpha\|\dot{p}_\varepsilon\|_{Q'/\ker B^*} \leq \sup_{v \,\varepsilon\, V} \frac{[p_\varepsilon,Bv]}{\|v\|_V}$$

$$= \sup_{v \,\varepsilon\, V} \frac{a(u_\varepsilon,v) - f(v)}{\|v\|_V}$$

$$\leq M(\|u_\varepsilon\|_V + \|f\|_{V'})$$

$$\leq C = \text{constant} \tag{3.16}$$

Thus, there exists a subsequence of functions p_ε, which converge weakly in $Q'/\ker B^*$ to a unique element p. One can easily show the weak limit (u,p) of $(u_{\varepsilon'},p_{\varepsilon'})$ is the unique solution of problem $(3.14)_1$.

<u>Strong Convergence</u>. When conditions N.1, N.2, and (3.15) hold, the situation is actually much better than that implied above. Indeed, subtracting $(3.14)_3$ from $(3.14)_1$, gives

$$a(u - u_\varepsilon,v) = [\dot{p} - p_\varepsilon,Bv] \quad \forall\, v \,\varepsilon\, V \tag{3.17}$$

Thus, from (3.15),

$$\alpha\|\overset{\bullet}{\overline{p - p_\varepsilon}}\|_{Q'/\ker B^*} \leq M\|u - u_\varepsilon\|_V . \tag{3.18}$$

Likewise, from $(2.5)_2$ and (3.17)

$$\begin{aligned} m\|u - u_\varepsilon\|_V^2 &\le a(u - u_\varepsilon, u - u_\varepsilon) \\ &= [\dot{p} - p_\varepsilon, Bu - Bu_\varepsilon] \\ &= \varepsilon[\dot{p} - p_\varepsilon, j^{-1}(p_\varepsilon)] \end{aligned} \tag{3.19}$$

On the other hand, inequalities in (3.9) implies

$$\|p_\varepsilon\|_{Q'}^2 \le \frac{2}{\varepsilon}\,(F(v_o) - F(u_\varepsilon))$$

for $v_o \in V$ such that $Bv_o = 0$. Since u_ε is uniformly bounded in ε in V, it follows from the inequality that

$$\|p_\varepsilon\|_{Q'} \le \frac{C}{\sqrt{\varepsilon}} \tag{3.20}$$

for a proper positive number $C>0$. Thus we have

$$\begin{aligned} m\|u-u_\varepsilon\|_V^2 &\le \varepsilon \|\overline{\dot{p}-p_\varepsilon}\|_{Q'/\ker B^*} \;\; \|p_\varepsilon\|_{Q'} \\ &\le C\sqrt{\varepsilon}\, \|\overline{\dot{p}-p_\varepsilon}\|_{Q'/\ker B^*}, \end{aligned} \tag{3.21}$$

since j^{-1} is an isometry. Combining (3.18) and (3.21), we have

$$\|u-u_\varepsilon\|_V \le C_1\sqrt{\varepsilon} \quad \text{and} \quad \|\overline{\dot{p}-p_\varepsilon}\|_{Q'/\ker B^*} \le C_2\sqrt{\varepsilon} \tag{3.22}$$

where C_1 and C_2 are positive numbers independent of ε. Thus, $u_\varepsilon \to u$ in V and $p_\varepsilon \to \dot{p}$ in $Q'/\ker B^*$ strongly as $\varepsilon \to 0$.

If $\ker B^* = \{0\}$ or if we use the norm $||\cdot||_{Q'/\ker B^*}^2$ in the definition of the perturbed Lagrangian instead of $||\cdot||_{Q'}^2$, then it is possible to obtain a rate of convergence of $O(\varepsilon)$ instead of $O(\sqrt{\varepsilon})$. For example, if $\ker B^* = \{0\}$, then (3.19) is changed to

$$\alpha||p-p_\varepsilon||_{Q'} \le M||u-u_\varepsilon||_V \tag{3.18*}$$

and

$$m\|u-u_\varepsilon\|_V^2 \le \varepsilon[p-p_\varepsilon, j^{-1}(p_\varepsilon)]$$

$$\le \varepsilon[p-p_\varepsilon, j^{-1}(p)] \qquad (3.19)^*$$

Then we have

$$\|u-u_\varepsilon\|_V \le C_1\varepsilon \quad \text{and} \quad \|p-p_\varepsilon\|_{Q'} \le C_2\varepsilon \qquad (3.22)^*$$

instead of (3.22).

3.4 Penalty Methods. In view of $(3.14)_4$, the perturbed multiplier can be eliminated from $(3.14)_3$ to give the variational problem

$$a(u\ ,v) + \varepsilon^{-1}[j(Bu_\varepsilon - g_o), Bv] = f(v) \quad \forall\, v \in V \qquad (3.23)$$

This is precisely the characterization of the minimizer of the penalty functional $F_\varepsilon : V \to \mathbb{R}$

$$F_\varepsilon(v) = F(v) + \frac{1}{2\varepsilon}\|Bv-g_o\|_Q^2 \qquad (3.24)$$

Thus, the perturbed Lagrange method is completely equivalent to the following (exterior) penalty method:

i) For each $\varepsilon > 0$, find a minimizer of F_ε over all of V

$$F_\varepsilon(u_\varepsilon) \le F_\varepsilon(v) \quad \forall\, v \in V \qquad (3.25)$$

ii) Under conditions N.1 and N.2, we will have a unique solution u_ε of (3.25) for every $\varepsilon > 0$ and, moreover, $u_\varepsilon \to u$ as $\varepsilon \to 0$, where u is the unique solution to the minimization problem (2.1)

iii) To obtain an approximation of the Lagrange multiplier p , take $p_\varepsilon = -\frac{1}{\varepsilon} j\ (Bu_\varepsilon - g_o)$.

iv) If the Babuska-Brezzi condition (3.15) holds, then we are assured that subsequences $p_\varepsilon \to \dot{p}$ in $Q'/\ker B^*$ as $\varepsilon \to 0$.

Similar formulations to (3.23) can be obtained for problems involving constraints represented by inequalities. Let B be a linear continuous operator from V into Q , and let g_o be a given data in Q . Using the partial ordering relation "$\leq$" defined on the space Q , the constrained set K is supposed to be given by

$$K = \{v \in V:\ Bv - g_o \leq 0\} \tag{3.26}$$

Then we have

(Lagrangian Multiplier Method)

$(u,p) \in V \times M$:

$$\begin{aligned} &\langle DF(u),v\rangle - [p,Bv] = 0 , \quad \forall\, v \in V \\ &[q-p, Bu-g_o] \geq 0 , \quad \forall\, g \in M \end{aligned} \tag{3.27}$$

(Perturbed Lagrange Multiplier Method)

$(u_\varepsilon, p_\varepsilon) \in V \times M$:

$$\begin{aligned} &\langle DF(u_\varepsilon),v\rangle - [p_\varepsilon, Bv] = 0 , \quad \forall\, v \in V \\ &[q-p_\varepsilon,\ \varepsilon j^{-1}(p_\varepsilon) + Bu_\varepsilon - g_o] \geq 0 , \quad \forall\, q \in M \end{aligned} \tag{3.28}$$

(Penalty Method)

$u_\varepsilon \in V$:

$$\langle DF(u_\varepsilon),v\rangle + \frac{1}{\varepsilon}[j(Bu_\varepsilon - g_o)^+, Bv] = 0 , \quad \forall\, v \in V \tag{3.29}$$

Here $\langle DF(u_\varepsilon),v\rangle = a(u_\varepsilon,v) - f(v)$ and "+" is the generalization in the function space Q of the operation defined on $\mathbb{R}$: $\phi^+ = (\phi + |\phi|)/2$ for $\phi \in \mathbb{R}$, and the set M is given by

$$M = \{q \in Q' : q \leq 0\} \tag{3.30}$$

We note that the penalty functional $F_\varepsilon : V \to \mathbb{R}$ is now defined by

$$F_\varepsilon(v) = F(v) + \frac{1}{2\varepsilon}\|(Bv-g_o)^+\|_Q^2 \tag{3.31}$$

4. PENALTY METHODS AND REDUCED INTEGRATION

Of all the variational methods discussed in the previous section, the penalty method is, perhaps, the most attractive as a basis for computational methods for the following reasons:

1) It involves minimization of a functional F_ε on the entire linear space V rather than on the constraint set K . Thus, it is not necessary to construct special approximations of the constraints.

2) For each $\varepsilon > 0$, only u_ε is unknown; approximations p_ε of Lagrange multipliers can be obtained by an independent and direct calculation, $p_\varepsilon = -\varepsilon^{-1}j(Bu_\varepsilon - g_o)$. Thus, the number of unknowns in a discretization of the penalty formulation is substantially less than a saddle-point formulation based on Lagrange multipliers.

3) The solutions $(u_\varepsilon,p_\varepsilon)$ to the penalty problem will converge to a solution (u,p) of the saddle point problem as $\varepsilon \to 0$ under the

conditions listed in the previous section. However, the penalty method will always yield a unique solution p_ε for the approximate multipliers while the direct saddle-point approximation of $\dot{p}$ may not be unique.

There are, however, some major difficulties in applying the penalty method due to the requirement (3.15). To demonstrate the key ideas, let us again consider the problem of incompressible elasticity. In this case, the penalized functional is

$$F_\varepsilon(\underset{\sim}{v}) = \frac{1}{2}a(\underset{\sim}{v},\underset{\sim}{v}) - f(\underset{\sim}{v}) + \frac{1}{2\varepsilon}\|\text{div } \underset{\sim}{v}\|_0^2 \tag{4.1}$$

where $a(\cdot,\cdot)$ is defined by (3.6) and $\|\cdot\|_0$ is the L^2-norm $(\|v\|_0^2 = \int_\Omega v^2\, dx)$. The associated variational boundary-value problem is then to find $u_\varepsilon \in V$ such that

$$a(\underset{\sim}{u}_\varepsilon,\underset{\sim}{v}) + \varepsilon^{-1}(\text{div } \underset{\sim}{u}_\varepsilon,\text{div } \underset{\sim}{v}) = f(\underset{\sim}{v}) \quad \forall\, \underset{\sim}{v} \in V \tag{4.2}$$

To construct a finite element approximation of (4.2), we develop, in the usual way, a family $\{V_h\}$ of finite-dimensional subspaces of V using C^0-piecewise polynomial basis functions on a sequence of meshes. The mesh size h identifies a space $V_h \subset V$ obtained through regular, uniform refinements of the mesh. We then approximate (4.2) on V_h with one provision: In anticipation of some difficulties to be described below, we shall evaluate the penalty term $(2\varepsilon)^{-1}\|\text{div } \underset{\sim}{v}\|_0^2$ using numerical quadrature. In particular, let $I(\cdot)$ denote the quadrature rule (for a continuous function f)

$$I(f) = \sum_{e=1}^{E} I_e(f) \; ; \quad I_e(f) = \sum_{j=1}^{G} W_j^e f(\xi_j^e) \tag{4.3}$$

where E is the number of finite elements, W_j^e the quadrature weights, and ξ_j^e the integration points in element $\Omega_e \subset \Omega$. Then approximation of the penalty method (4.2) consists of seeking $\underset{\sim}{u}_\varepsilon^h \in V_h$ such that

$$a(\underset{\sim}{u}_\varepsilon^h, \underset{\sim}{v}^h) + \varepsilon^{-1} I(\text{div}\ \underset{\sim}{u}_\varepsilon^h\ \text{div}\ \underset{\sim}{v}^h) = f(\underset{\sim}{v}^h) \quad \forall\ \underset{\sim}{v}^h \in V_h \qquad (4.4)$$

Of course, if the order G of the integration rule (4.3) is sufficiently large, we can expect $I(\cdot)$ to yield exact integration: $I(fg) = (f,g) = \int_\Omega fg\ dx$.

It is a remarkable fact that (4.4), in general, provides physically unacceptable numerical results under certain boundary conditions if exact integration is used. This has led many investigators to use "reduced integration" for finite element methods based on penalty formulations - by which is meant the use of an integration rule $I(\cdot)$ of order G lower than that which is necessary to integrate the penalty terms exactly. Such devices have been advocated by Zienkiewicz, Taylor, and Too[16], Hughes[8], Malkus and Hughes[12], and others; for a complete analysis of such methods and additional references, see Oden, Kikuchi, and Song[14,15].

A key to understanding penalty methods and reduced integration is the realization that, for fixed mesh size h, the solution to the discrete penalty problem should converge, in some sense, to a mixed finite element approximation of (3.5). Thus, there must be inherent in any discrete penalty method, a finite-dimensional space Q_h' approximating the space Q' of Lagrange multipliers. With this in mind, we choose the space Q_h to satisfy the following three conditions for reduced-integration/penalty (RIP) methods:

R.1 Q_h is such that $\forall\, q^h \in Q_h'$ and $\forall\, \underset{\sim}{v}^h \in V_h$,

$$(q^h, \operatorname{div} \underset{\sim}{v}^h) \;=\; I(q^h \operatorname{div} \underset{\sim}{v}^h) \tag{4.5}$$

R.2 Let $B_h : V_h \to Q_h$, $B_h^* : Q_h' \to V_h'$ be defined by

$$I(q^h \operatorname{div} \underset{\sim}{v}^h) \;=\; [q^h, B_h \underset{\sim}{v}^h] \;=\; \langle B_h^* q^h, \underset{\sim}{v}^h \rangle \tag{4.6}$$

(i.e. B_h is the discrete approximation of the operator div defined by our choice of integration rule I). Then Q_h' must have the property that constants $\alpha_h > 0$ exist such that the following discrete Babuska-Brezzi conditions hold:

$$\alpha_h \|\overset{o}{q}{}^h\|_{0/\ker B_h^*} \leq \sup_{\underset{\sim}{v}^h \in V_h} \frac{I(q^h \operatorname{div} \underset{\sim}{v}^h)}{\|\underset{\sim}{v}^h\|_V} \quad \forall\, q^h \in Q_h' \tag{4.7}$$

R.3 There exists a unique $p_\varepsilon^h \in Q_h'$ such that

$$p_\varepsilon^h(\underset{\sim}{\xi}{}_j^e) \;=\; -\varepsilon^{-1} \operatorname{div} \underset{\sim}{u}{}_\varepsilon^h(\underset{\sim}{\xi}{}_j^e) \tag{4.8}$$

for $1 \leq e \leq E$, $1 \leq j \leq G$ (effectively, the nodal points of elements defining Q_h are the integration points given in the definition of $I(\cdot)$), and the notation

$$\|\overset{o}{q}{}^h\|_{0/\ker B_h^*} \;=\; \inf_{(q^h)^* \in \ker B_h^*} \|q^h + (q^h)^*\|_0$$

is applied where $\ker B_h^*$ is the kernel of the map B_h^*. If $\ker B_h^* \subset \ker B^*$, we have

$$\|\overset{o}{q}{}^h\|_{0/\ker B_h^*} = \|\dot{q}^h\|_{0/\ker B^*},$$

and $\overset{o}{q}{}^h$ may be identified with the coset $\dot{q}^h$ with respect to the kernel of B^*.

Condition R.1 does not hold for all reduced - integration - penalty methods. In some instances, where, for example, quadratic polynomials are higher are used for the velocity but only 1 - point integration is used for the penalty, this condition does not hold and a loss of accuracy due to quadrature results. Nevertheless, conditions R.2 and R.3 are imposed.

Oden, Kikuchi, and Song[14] have shown that if conditions R.1, R.2, and R.3 hold, then the following can be concluded:

i) For every $\varepsilon > 0$ and $h > 0$, there exists a unique solution $\underset{\sim}{u}{}_\varepsilon^h$ to (4.4)

ii) The sequence $\{(\underset{\sim}{u}{}_\varepsilon^h, p_\varepsilon^h)\}$, where p_ε^h is given by (4.9), converges to $(\underset{\sim}{u}{}^h, \overset{o}{p}{}^h)$ as $\varepsilon \to 0$ for fixed h, where $(u^h, \overset{o}{p}{}^h)$ is a solution of the mixed finite element approximation:

$$a(\underset{\sim}{u}{}^h, \underset{\sim}{v}{}^h) - I(\overset{o}{p}{}^h \operatorname{div} \underset{\sim}{v}{}^h) = f(\underset{\sim}{v}{}^h) \quad \forall\ \underset{\sim}{v}{}^h \in V_h$$

$$I(q^h \operatorname{div} \underset{\sim}{u}{}^h) = 0 \quad \forall\ q^h \in Q_h' \tag{4.9}$$

iii) For fixed $\varepsilon > 0$ and, $h > 0$, the following error estimates hold.

$$||\underset{\sim}{u} - \underset{\sim}{u}^h_\varepsilon||_1 \leq C_1 \; (1 + \alpha_h^{-1}) \; \Gamma_h \; (\underset{\sim}{u},p)$$

$$||\overset{o}{p} - \overset{o}{q}^h||_{0/\ker B_h^*} \leq C_2 \; (1 + \alpha_h^{-1} + \alpha_h^{-2}) \; \Gamma_h \; (\underset{\sim}{u},p) \tag{4.10}$$

$$\Gamma_h \; (\underset{\sim}{u},p) = \inf_{\underset{\sim}{v}^h \varepsilon V_h} ||\underset{\sim}{u} - \underset{\sim}{v}^h||_1 + \sqrt{\varepsilon}$$

$$+ \inf_{q^h \varepsilon Q_h'} || \dot{p} - \dot{q}^h ||_{0/\ker B^*}$$

where C_1, C_2 are constants independent of ε and h. If $\ker B_h^* = \{0\}$, then $\sqrt{\varepsilon}$ can be replaced by ε.

iv) According to (4.9), every RIP-method is related to a mixed method. However, there is one major difference: the direct approximation (4.9) may not have a unique solution! On the other hand, by construction, a unique $(u^h_\varepsilon, p^h_\varepsilon)$ solution to (4.9) is obtained for each $\varepsilon > 0$. The point here is that, in general, $\ker B_h^* \not\subset \ker B^*$ and $\ker B_h^* \neq \{0\}$, particularly for rectangular elements. Thus, the conventional mixed finite element method may produce approximate multipliers p^h with components in $\ker B_h^*$. Then the system (4. 9) is singular! However, any RIP-method satisfying R.1-R.3 will have

$$p_\varepsilon^h(\underset{\sim}{\xi}_j^e) = -\varepsilon^{-1} \operatorname{div} \underset{\sim}{u}_\varepsilon^h(\underset{\sim}{\xi}_j^e) \tag{4.11}$$

Thus, let $q_o^h \in \ker B_h^*$; i.e.

$$I(q_o^h \operatorname{div} \underset{\sim}{v}^h) = \langle B^* q_o^h, \underset{\sim}{v}^h \rangle = 0 \quad \forall \underset{\sim}{v}^h \in V_h$$

Then, from

$$I(p_\varepsilon^h q_o^h) = 0 \quad \forall q_o^h \in \ker B_h^* \tag{4.12}$$

In other words, p_ε^h is orthogonal to $\ker B_h^*$ with respect to the discrete inner product $I(fg)$. Indeed, if (4.5) holds, this orthogonality is with respect the $L^2(\Omega)$-inner product .

5. A PATCH TEST FOR RIP-METHODS

The numerical stability of the RIP-method is governed by the discrete Babuska-Brezzi condition (4.7). If it is not satisfied, there may occur oscillations in the approximate pressure p_ε^h as $h \to 0$ which quickly become unbounded. It is obvious that a necessary condition for (4.7) to hold is that the matrix $\underset{\sim}{B}$ approximating the constraint of full rank, i.e., if

$$\dim V_h = n , \quad \dim Q_h = m , \quad (q^h, \operatorname{div} \underset{\sim}{v}^h) = \sum_{r,s=1}^{n,m} B^{rs} q_r v_s$$

where q_r, v_s are the degrees-of-freedom of the approximate pressures and displacements, respectively. Then we must have

$$\operatorname{rank}(B^{rs}) = m \tag{5.1}$$

if (4.7) is to hold.

However, (5.1) is not enough! For an unconditionally stable method, we must generally have α_h in (4.7) independent of h.

We shall now describe a method for testing the LBB-condition for specific choices of elements and integration rules that is based on some techniques of Girault and Raviart[7] for mixed methods. The idea is to find a function $\hat{\underset{\sim}{v}}^h$ in the finite element space V_h such that, for any $\underset{\sim}{v} \in V$,

$$\int_\Omega \operatorname{div}(\hat{\underset{\sim}{v}}^h - \underset{\sim}{v})\, q^h\, dx = 0 \quad \forall\, q^h \in Q_h' \tag{5.2}$$

$$\|\hat{\underset{\sim}{v}}^h\|_V \leq C\|\underset{\sim}{v}\|_V$$

Then it can be shown (cf. Oden, Kikuchi, and Song[14]) that whenever (5.2) holds, the discrete LBB-condition (4.7) is satisfied with a constant α_h independent of h.

From these facts, we can construct a "patch test" for verifying the stability of RIP-methods:

1. Pick an arbitrary $\underset{\sim}{w}^h \in V_h$.

2. Construct the patch integral

$$I_p = \int_{\Omega_h} \operatorname{div}\,(\underset{\sim}{v}^h - \underset{\sim}{w}^h) q^h\, dx = - \int_{\Omega_h} \nabla q^h \cdot (\underset{\sim}{v}^h - \underset{\sim}{w}^h)\, dx$$

$$+ \sum_e \int_{\partial\Omega_e} \underset{\sim}{n} \cdot (\underset{\sim}{v}^h - \underset{\sim}{w}^h)\, q^h\, ds$$

for a patch Ω_h of elements with interelement boundaries $\partial\Omega_e$, ($\underset{\sim}{n}$ a unit outward normal to $\partial\Omega_e$) with $q^h \notin \ker B_h^*$.

3. Determine, by direct calculation, if there exists a v^h such that $I_P = 0$ and $\|\underset{\sim}{v}^h\|_V \leq C\|\underset{\sim}{w}^h\|_V$.

4. For arbitrary $v \in V$, set $\underset{\sim}{w}^h = P_h \underset{\sim}{v}$, where P_h is the V-orthogonal projection of V onto V_h . Determine C , independent of h , such that $I_P = 0$ and $\|\underset{\sim}{v}^h\|_V \leq C\|\underset{\sim}{v}\|_V$.

If these four steps prove to be possible for a given choice of V_h and $I(\cdot)$, then (4.7) holds for an α_h independent of h .

There are actually only a limited number of elements which have been shown to satisfy such patch tests. For two-dimensional problems with smooth domains and Dirichlet boundary conditions, the following choices satisfy the discrete LBB-condition:

	V_h	I	Q_h
1.	6-node, quadratic triangles	1-point Gaussian quadrature	Piecewise constants
2.	9-node, biquadratic rectangle	"1-point" integration	Piecewise constants/integration
3.	8-node, seredipidity element	"1-point" integration	Piecewise constants/integration
4.	Composite of four equal 4-node bilinear elements	"3-point" integration	Piecewise linear
5.	9-node-biquadratic	"3-point" integration	Piecewise linear

In these examples, $\ker B_h^* \subset \ker B^*$ and the analysis is based on a uniform mesh on a rectangular domain. By "1-point" and "3-point" integration in the examples, we mean that the method is equivalent to some

1-point and 3-point rule, but that the actual calculations are done using a perturbed-Lagrangian formulation; e.g. for a typical element we have

$$\underset{\sim}{K}\,\underset{\sim}{u}_{\varepsilon} - \underset{\sim}{B}\,\underset{\sim}{p}_{\varepsilon} = \underset{\sim}{f} + \underset{\sim}{\sigma}$$

$$\underset{\sim}{B}^{T}\,\underset{\sim}{u}_{\varepsilon} + \varepsilon\,\underset{\sim}{M}\,\underset{\sim}{p}_{\varepsilon} = \underset{\sim}{0} \qquad (5.3)$$

where $\underset{\sim}{K}$ and $\underset{\sim}{f}$ are the element stiffness matrix and load vector $\underset{\sim}{\sigma}$ the "connecting vector" of forces (which vanish upon assembling the elements) and $\underset{\sim}{B}$ is the "constraint matrix". In (5.3), $\underset{\sim}{M}$ is the mass matrix corresponding to the local pressure approximations assumed here to be discontinuous across interelement boundaries. If we use piecewise constant or linear approximations for $\underset{\sim}{p}_{\varepsilon}$, we then calculate

$$\underset{\sim}{p}_{\varepsilon} = -\varepsilon^{-1}\underset{\sim}{M}^{-1}\underset{\sim}{B}^{T}\underset{\sim}{u}_{\varepsilon} \qquad (5.4)$$

Then $(5.3)_1$ results in a penalty method

$$\underset{\sim}{K}\,\underset{\sim}{u}_{\varepsilon} + \varepsilon^{-1}\underset{\sim}{B}\,\underset{\sim}{M}^{-1}\underset{\sim}{B}^{T}\underset{\sim}{u}_{\varepsilon} = \underset{\sim}{f} + \underset{\sim}{\sigma}$$

even though no integration scheme has actually been identified. It is such schemes that are understood to be used in the examples above.

Numerical results indicate that the Babuska-Brezzi constant α_h for schemes 1-5 is independent of h :

$$\alpha_h = \alpha_o = \text{constant} > 0$$

Then, these schemes are numerically stable. However, there is a price to be paid for schemes 1-3: <u>the low integration rule leads to a loss of one complete order in the asymptotic rates of convergence of these methods</u>.

Thus, while stable, they converge slowly. Schemes 4 and 5 appear to be much better and are theoretically sound, but their analyses remains incomplete.

We mention two other examples which are popular in the literature:

	V_h	I	Q_h
6.	4-node bilinear	1-point Gaussian	Piecewise constant
7.	9-node bilinear	2-point Gaussian quadrature	Piecewise bilinear

In both of these cases (see [15]), we obtain

$$\alpha_h = \alpha_o h = O(h)$$

for mixed boundary conditions and

$$\alpha_h = O(h^2);\ \ker B_h^* \not\subset \ker B^*$$

for Dirichlet boundary conditions.

Thus, these methods might be unstable. The convergence of such methods for rectangular uniform meshes has been explained in analyses by Johnson and Pitkaranta[8] and in an independent study by the authors[11] using a different approach. Nevertheless, the pressure may diverge in $L^2(\Omega)$ (due largely to the presence of spurious components in $\ker B_h^*$). Likewise, method 7 may have a suboptimal rate of convergence for displacements in V and lead to pressure approximations which diverge in $L^2(\Omega)$.

It is true, however, that "filtering" schemes can be devised for these elements which produce pressure approximations which are stable and converge in $L^2(\Omega)$. According to the results in 14 scheme 4 might be interpreted as the filtering strategy for scheme 6. However, method 6 and 7 still appear to be delicate: they are sensitive to singularities and distortions in the mesh geometry. Despite their popularity, these observations indicate that they should be used with great care.

6. CHECKERBOARDS AND NUMBERS

Several authors (e.g. Lee et al[11]) have observed that in mixed finite element approximations of certain constrained problems, particularly Stokes problems or incompressible elasticity problems with constraints such as $\operatorname{div} \underset{\sim}{u} = 0$, spurious checkerboard patterns for the pressures are present which appear to be superimposed over solutions which are physically reasonable. These spurious patterns are explained without difficulty in the theory outlined previously for the case of square uniform meshes.

Observe that for RIP methods, the approximation p_ε^h of the hydrostatic pressure may include components in $\ker B_h^*$, even though it is clear that $B_h^* p_\varepsilon^h \neq 0$.

Let us attempt to characterize $\ker B_h^*$ for a representative finite element for which spurious nodes have been detected. In particular, in the case of a square domain on which Dirichlet boundary conditions have been imposed, we consider a uniform mesh of 4-node bilinear elements for approximating the displacement $\underset{\sim}{v}^h$ and piecewise constants for the pressures q^h. This corresponds to 1-point Gauss integration for the rule $I(\cdot)$ in the RIP-method discussed in Section 4. For a mesh of N elements of width h, it can be shown ([15])

$$(q^h, \operatorname{div} v^h) = 2h \sum_{k=1}^{N} [V_{k1}(q_k^1 + q_k^2 - q_k^3 - q_k^4)$$

$$+ V_{k2}(q_k^1 - q_k^2 - q_k^3 + q_k^4)] \qquad (6.1)$$

where V_{ki} is the i-th component of $\underset{\sim}{v}^h$ at node k (i = 1,2), and q_k^m is the value of the (constant) pressure q^h in cell m surrounding node k, m = 1,2,3,4, with m = 1 for the lower left quadrant and the rest numbered in counterclockwise sequence around node k. Let us make the identification, $Q_h = Q_h'$. If $q^h \notin \ker B_h^*$, then q^h is in the range of B_h and one can find a $\underset{\sim}{v}^h \in V_h$ such that $B_h \underset{\sim}{v}^h = q^h$. Hence, if $q^h \in \ker B_h^*$, we must have

$$q_k^1 + q_k^2 - q_k^3 - q_k^4 = 0 \;, \quad q_k^1 - q_k^2 - q_k^3 + q_k^4 = 0$$

This is precisely the checkerboard pattern indicated in Fig. 1. Thus, the spurious modes observed in mixed finite-element calculations are merely the components of the approximate pressure in $\ker B_h^*$. These spurious components should not appear in any RIP method for which condition R.1, R.2, and R.3 of Section 4 hold, provided there are no singularities in the solution. However, as will be shown in examples, spurious components are sometimes observed in the pressure p_ε^h <u>locally</u>, e.g. in certain subsets of of the mesh. This generally occurs in methods for which $\alpha_h = O(h^\sigma)$, $\sigma > 0$, in which case the method may unstable in $L^2(\Omega)$.

Spurious components in p_ε^h by e.g. the scheme 6 (4-node bilinear and and 1-point integration) can be eliminated by taking the filtering operation used in convergence analysis and also introduced in the work[11]. Indeed, the component of "local" $\ker B_h^*$ is defined in a composite of four equal 4-node bilinear elements as

$$C = (p_\varepsilon^1 - p_\varepsilon^2 + p_\varepsilon^3 - p_\varepsilon^4) \ , \tag{6.2}$$

where $p_\varepsilon^i = p_\varepsilon^h|_{\Omega_e^i}$, $1 \leq i \leq 4$, and Ω_e^i is the i-th 4-node element within the e-th composite element. Then the pressure p_ε^h is filtered by $\tilde{p}_\varepsilon^h$ defined by

$$\tilde{p}_\varepsilon^1 = p_\varepsilon^1 - C \ , \quad \tilde{p}_\varepsilon^2 = p_\varepsilon^2 + C \ , \quad \tilde{p}_\varepsilon^3 = p_\varepsilon^3 - C \ , \quad \tilde{p}_\varepsilon^4 = p_\varepsilon^4 + C \tag{6.3}$$

within each composite element. The pressure $\tilde{p}_\varepsilon^h$ is then with linear function which corresponds to the case of scheme 4. As shown in the following example, $\tilde{p}_\varepsilon^h$ does not include any spurious mode any more.

Example: One example involves a Dirichlet problem of plane strain of a square slab of incompressible linearly elastic material subjected to a constant body force $f = (800,800)$ applied over a square domain Ω_0 . Let Young's modulus be $E = 10^3$. We use the rather coarse mesh of 16 elements shown in Fig. 2 and employ

Q_2 - elements (9-Node Biquadratics) for V_h

$I(\cdot)$ ~ 2 × 2 - Gaussian Quadrature

Figure 3 shows computed hydrostatic pressure along section A-A.

For other choice of data, we observe the checkerboard modes in hydrostatic pressure. Indeed, if a point load $f = 200(\delta(x-\bar{x},y-\bar{y}), \delta(x-\bar{x},y-\bar{y}))$ is applied at point $(\bar{x},\bar{y}) \in \Omega$, then the checkerboard modes in $\ker B_h^*$ appear to be activated as numerical results in Fig. 4 shows.

7. CONTACT PROBLEMS IN ELASTICITY

Results similar to those described above can be obtained for problems with inequality constraints such as that encountered in contact problems in linear elasticity. Let us return to problem (2.14) with the constraint set K being given by the set K_2 of (2.12):

$$K_2 = \{\underset{\sim}{v} \in V \mid \underset{\sim}{v} \cdot \underset{\sim}{n} - s \leq 0 \quad \text{a.e. on } \Gamma_C\} \tag{7.1}$$

Then the contact problem for compressible materials can be expressed as the variational inequality

$$\underset{\sim}{u} \in K_2 : \quad a(\underset{\sim}{u},\underset{\sim}{v}-\underset{\sim}{u}) \geq f(\underset{\sim}{v}-\underset{\sim}{u}) \quad \forall\, \underset{\sim}{v} \in K_2 \tag{7.2}$$

whereas its corresponding penalty formulation is

$$\underset{\sim}{u}_\varepsilon \in V : \quad B(\underset{\sim}{u}_\varepsilon,\underset{\sim}{v}) + \frac{1}{\varepsilon}\langle (j(\underset{\sim}{u}_\varepsilon)_n - s)^+, v_n\rangle = f(\underset{\sim}{v}) \quad \forall\, \underset{\sim}{v} \in V \tag{7.3}$$

where $\langle\cdot,\cdot\rangle$ denotes duality pairing on $W' \times W$, where W is the Sobolev space

$$W = H^{1/2}(\Gamma_C)$$

$u_n \equiv u \cdot n$ (n being a unit outward normal to Γ_C), and $(\phi)^+$ is the positive part of the function ϕ (e.g. $\phi^+ = \max(0,\phi)$ relative to an ordering defined by $\phi \leq 0 \Rightarrow \gamma(x) \leq 0$ a.e. on Γ_C).

To approximate (7.3), we use the RIP-method:

$$\underset{\sim}{u}^h \in V_h : \quad B(\underset{\sim}{u}^h,\underset{\sim}{v}^h) + \frac{1}{\varepsilon}J[((\underset{\sim}{u}_\varepsilon^h)_n - s)^+ v_n^h] = f(\underset{\sim}{v}^h) \quad \forall\, \underset{\sim}{v}^h \in V_h \tag{7.4}$$

where V_h is, as before, a finite-dimensional subspace of V and J is a quadrature rule for integration Γ_C ; e.g.

$$J(f) = \sum_{e=1}^{E'} \sum_{j=1}^{G'} Q_j^e\, f(\eta_j^e) \tag{7.5}$$

where Q_j^e are quadrature weights and η_j^e quadrature points on Γ_C .

The method is stable and convergent whenever J defines a finite-dimensional space W_h of approximate contact pressure σ_ε^h such that

<u>R.1</u> $\forall\ \tau^h, \hat{\tau}^h \in W_h$,

$$J(\tau^h \hat{\tau}^h) = \int_{\Gamma_C} \tau^h \hat{\tau}^h \, ds \tag{7.6}$$

<u>R.2</u> There exists a unique $\sigma_\varepsilon^h \in W_h$ such that

$$\sigma_\varepsilon^h(n_j) = -\frac{1}{\varepsilon}((\underset{\sim}{u}_\varepsilon^h)_n - s)\ (n_j)^+ \tag{7.7}$$

<u>R.3</u> There is a constant $\beta_0 > 0$, independent of h (as $h \to 0$) such that $\beta_h = \beta_0 h^{1/2}$ and

$$\beta_h \|\tau^h\|_{0,\Gamma C} \leq \sup_{v^h \in V_h} \frac{|J(\tau^h v_n^h)|}{\|v^h\|_1} \tag{7.8}$$

$$\forall\ \tau^h \in W_h .$$

Again, the discrete Babuska-Brezzi condition (7.8) is the key to numerical stability of the approximations of the contact pressures.

<u>Example</u>: As an example, we consider the plane strain problem of indentation of a rigid cylindrical punch into a rectangular block of incompressible linearly elastic foundation which has a modulus of elasticity of $E = 10^3$. We model half the domain with 4 × 8-inch mesh of rectangular elements, as shown in Fig. 5 .

Figure 6 shows results obtained using

Q_2-elements (9-Node Biquadratic) for V_h

I(·) - 2 × 2 Gaussian Quadrature

J(·) - Simpson's Rule

Computed results are quite satisfactory. However, if we use

J(•) - 2-Point Gaussian Rule

the distribution of contact pressure is no longer smooth, as shown in Fig. 11.

These unstable pressure distributions can again be explained by the LBB-condition (7.8). Indeed, for Simpson's rule, it is shown in [15] that

$$\beta_h = \beta_0 h^{1/2} > 0$$

However, for 2-point Gaussian rule, we have

$$\beta_h = \beta_0 h^{1/2} - \beta_1 h^{1/2}$$

Thus, this is not suitable for the Babusak-Brezzi condition (7.8).

Thus, for certain models, β_h may be negative, in which case the discrete problem is ill-posed.

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APPENDIX

Some Mathematical Preliminaries

This appendix is provided for a brief review of several of the mathematical ideas used throughout this paper. For additional details, see standard texts on functional analyses; e.g. TAYLOR and LAY[17].

i) Strong and Weak Convergence. The spaces V of admissible functions described earlier, we recall, are assumed to be real Hilbert spaces equipped with inner products $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$ (e.g., when V is given by (2.4), $(\underset{\sim}{u},\underset{\sim}{v})_V = (\underset{\sim}{u},\underset{\sim}{v})_1 = \int_\Omega u_{i,j}\, v_{i,j}\, dx$, $\|\underset{\sim}{u}\|_V = \|\underset{\sim}{u}\|_1 = (\underset{\sim}{u},\underset{\sim}{u})_1^{1/2}$) . When we say that a sequence $u_m \varepsilon V$ converges to an element $u \varepsilon V$, we mean that

$$\lim_{m \to \infty} \|u_m - u\|_V = 0$$

This type of convergence is called strong convergence; to be specific we may say that u_m converges strongly to u and we write

$$u_m \to u \text{ (as } m \to \infty)$$

For each space V one has its (topological) dual space V' which consists of all continuous linear functionals defined on V . Thus, V' consists of linear functions f which map V continuously into real numbers. As noted earlier, we use the notation

$$f(u) \equiv \langle f,u \rangle \quad f \varepsilon V' , \ u \varepsilon V$$

to denote the value of f at u .

Whenever a sequence $u_m \varepsilon V$ has the property that

$$\lim_{m \to \infty} f(u_m) = f(u) \quad \forall\, f \in V'$$

we say that u_m converges weakly to u and we write

$$u_m \rightharpoonup u \qquad (\text{as } m \to \infty)$$

Every strongly convergent sequence converges weakly, but the converse is not true. For example, consider the Fourier series representation of functions g in $L^2(0,1)$:

$$g(x) = \sum_{m=1}^{\infty} g_m \gamma_m(x); \quad \gamma_m(x) = \sqrt{2} \sin m\pi x; \quad 0 < x < 1$$

$$g_m = \langle g,\gamma_m \rangle \equiv \int_0^1 g\gamma_m \, dx$$

The sequence g_m of real numbers converges to zero as m tends to infinity. Hence, the sequence of functions γ_m converges weakly to zero:

$$\lim_{m \to \infty} \langle g,\gamma_m \rangle = \lim_{m \to \omega} g_m = 0 \quad \forall\, g \in L^2(0,1)$$

However, $\|\gamma\|_{L^2(0,1)}=1$ so that the γ_m does not converge strongly to zero.

ii) <u>Weak Compactness</u>. In Hilbert spaces, it can be shown that every sequence bounded in norm has a weakly convergent subsequence. In other words, if u_m is a sequence in V with the property that

$$\|u_m\|_V \leq C = \text{constant}$$

then we are guaranteed the existence of a subsequence $u_{m'}$ and an element $u \in V$ such that $u_{m'} \rightharpoonup u$.

iii) Riesz Map. The Riesz representation theorem establishes that for every linear functional f in the dual V' of a Hilbert space V, there exists a unique element u_f such that

$$\langle f,v\rangle = (u_f,v)_V \quad \forall\, v \in V$$

In other words, the action of any linear functional f on V can be reproduced by an inner product of the elements of V by an element u_f uniquely determined by every f.

The correspondence between f and u_f can be characterized by a mapping $j_V : V \to V'$ known as the Riesz map for V. The operator j_V is linear, continuous, invertible, and is an isometry from V onto V' (i.e. $\|u\|_V = \|j_V u\|_{V'}$).

iv) Quotient Space. Let M and N denote a linear subspaces of a linear vector space V such that $M \cap N = \{0\}$ and every $v \in V$ is of the form $v = m + n$, with $m \in M$ and $n \in N$. We then say that V is the direct sum of M and N, written $V = M + N$, and that M and N are complementary subspaces of V; in particular, N is a complement of M and vice versa.

We introduce an equivalence relation R on V according to which vectors v_1 and v_2 are said to be equivalent modulo M if $v_1 - v_2 \in M$. An equivalence class corresponding to R is then the set

$$v^{\cdot} = \{u \in V \mid v - u \in M\}$$

The set of all such equivalence classes is denoted V/M. When endowed with the operations $u^{\cdot} + v^{\cdot} = (u + v)^{\cdot}$ and $\alpha u^{\cdot} = (\alpha u)^{\cdot}$, V/M is a linear vector space, called the quotient space of V modulo M. If V is a Banach space with norm $\|\cdot\|_V$ then V/M is a Banach space when provided the norm.

$$\|v\cdot\|_{V/M} = \inf_{u \in M} \|v - u\|_V$$

If N is any complement of M , then V/M is isomorphic to N (indeed, the map $\gamma : V \to V/M$, $\gamma(v) = v\cdot$, is an isomorphism, If M is a closed linear subspace of a Banach space V , then the dual space $(V/M)'$ is isometrically isomorphic to $M^{\perp}$, where $M^{\perp} = \{v^* \in V' | v^*(v) = 0 \text{ if } v \in M\}$. Thus, $(V/M)'$ and $M^{\perp}$ are both algebraically and metrically (and, therefore, topologically) equivalent, and we write $(V/M)' \approx M^{\perp}$.

LIST OF FIGURES

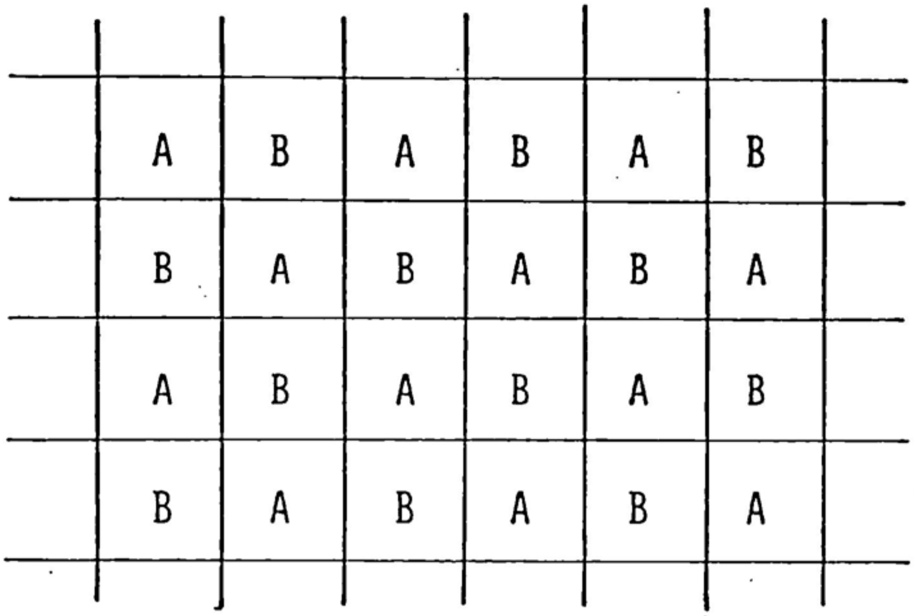

FIGURE 1. Kernel of B_h^* .

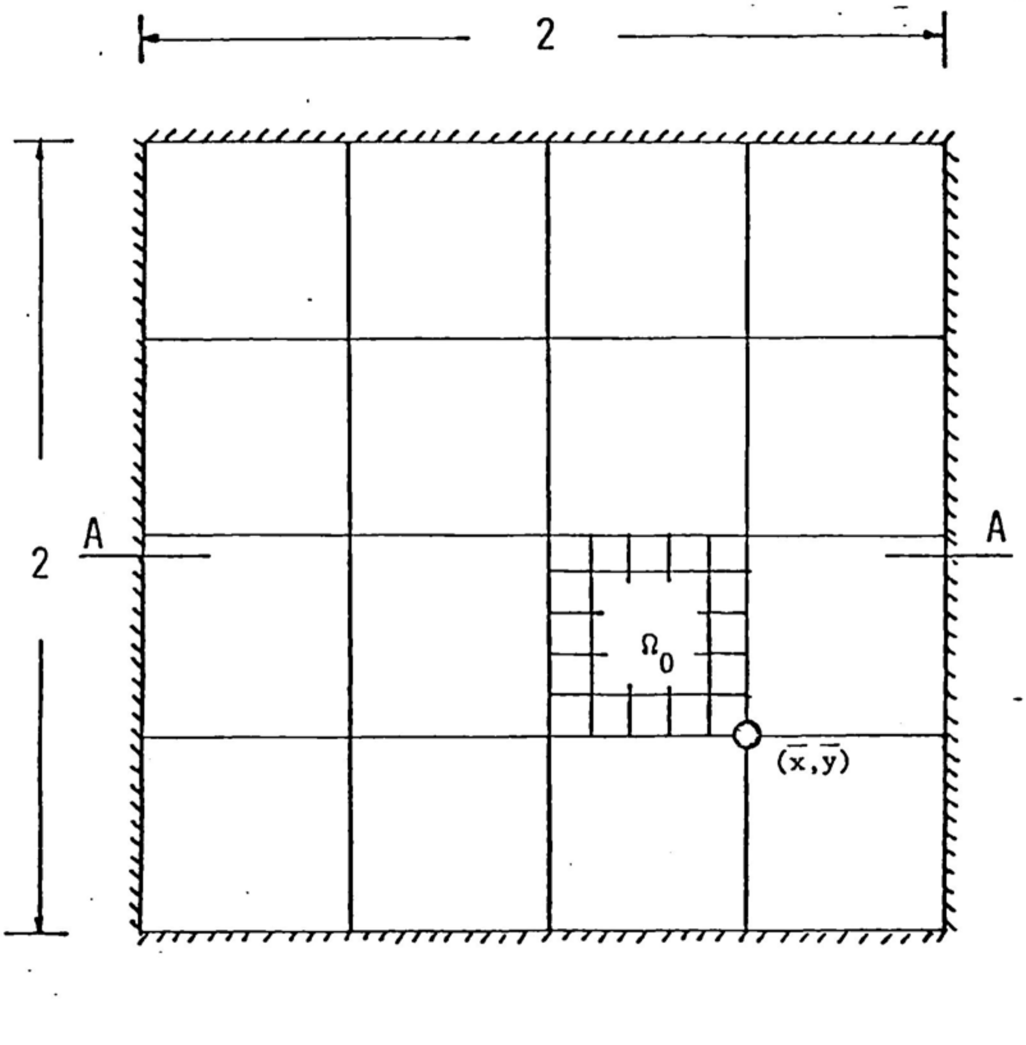

FIGURE 2. Finite Element Model of a Dirichlet Problem.

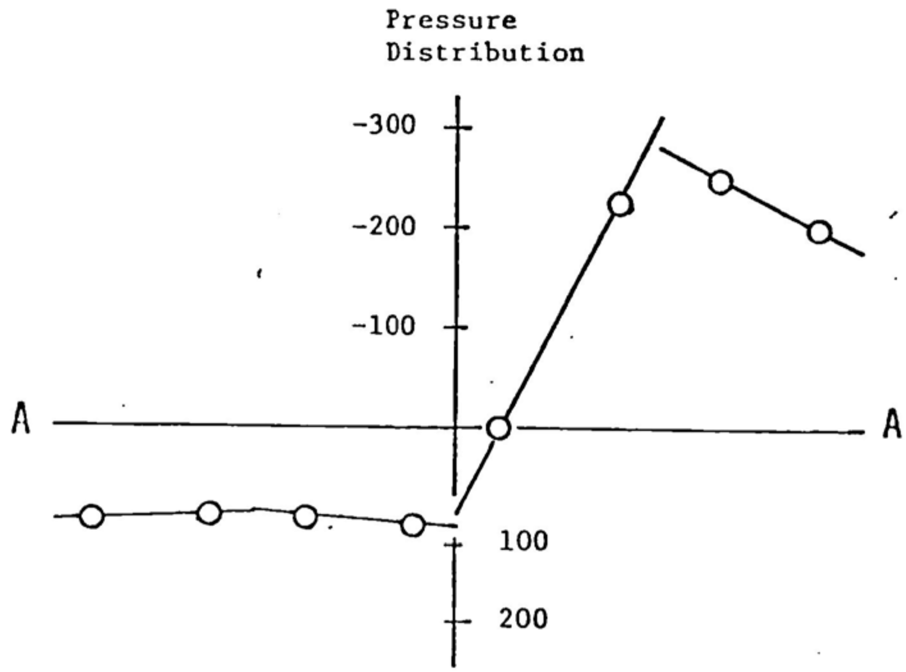

FIGURE 3. Pressure Distribution on the Cross Section A-A by 2 × 2 Gaussian Rule for a Smooth Applied Body Force.

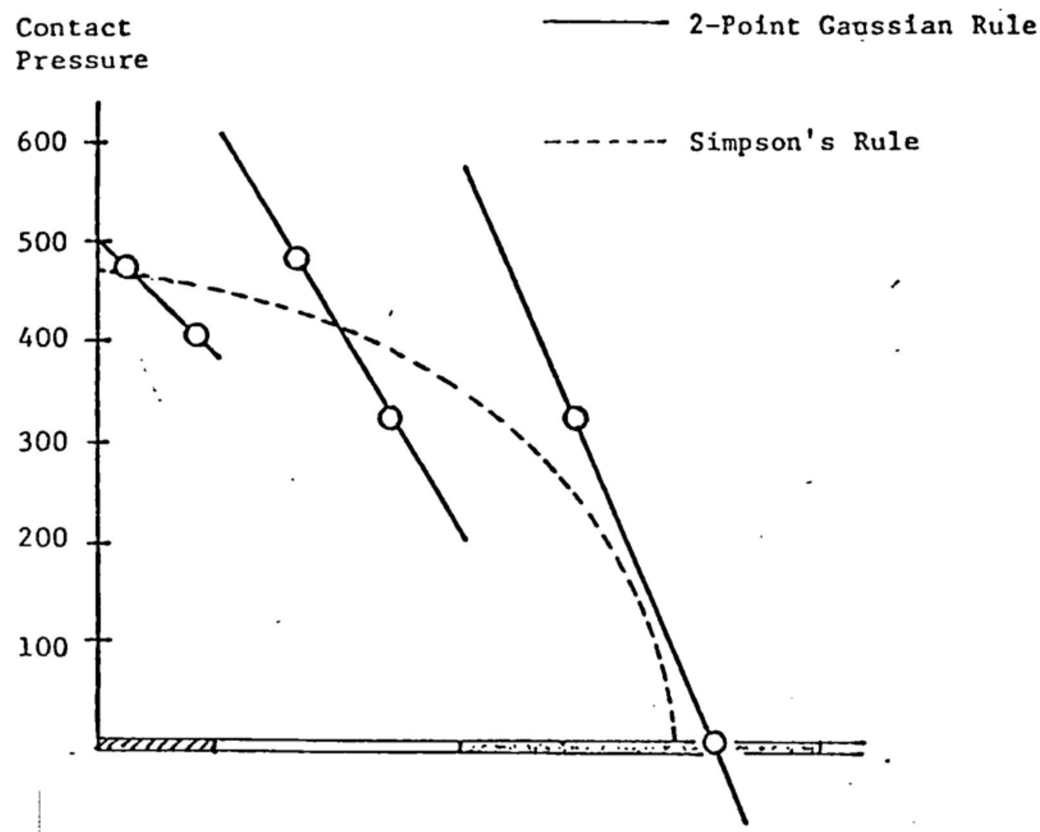

FIGURE 7. Pressure Distribution by 2-Point Gaussian Rule.

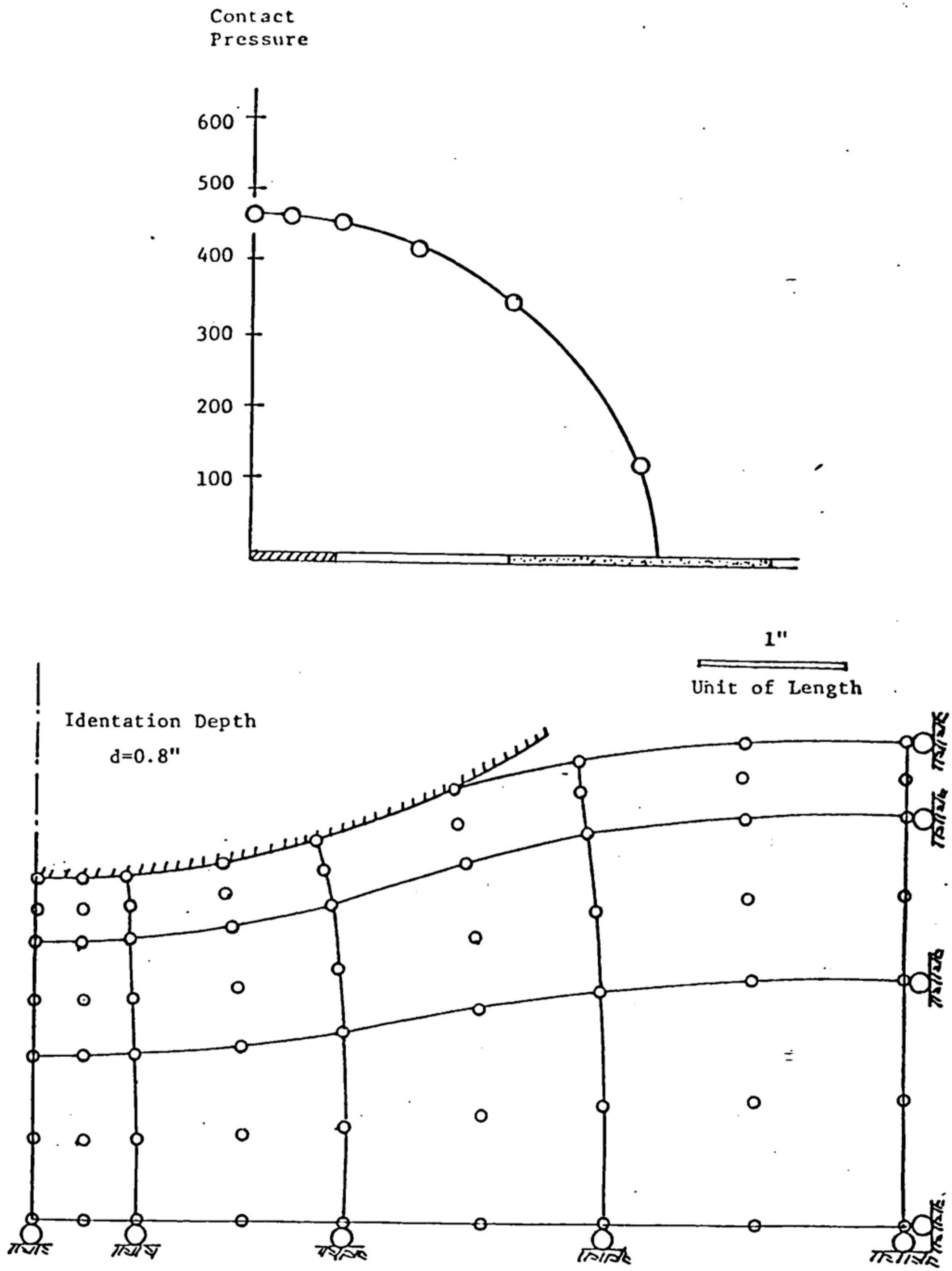

FIGURE 6 . Numerical Results by Simpson's Rule.

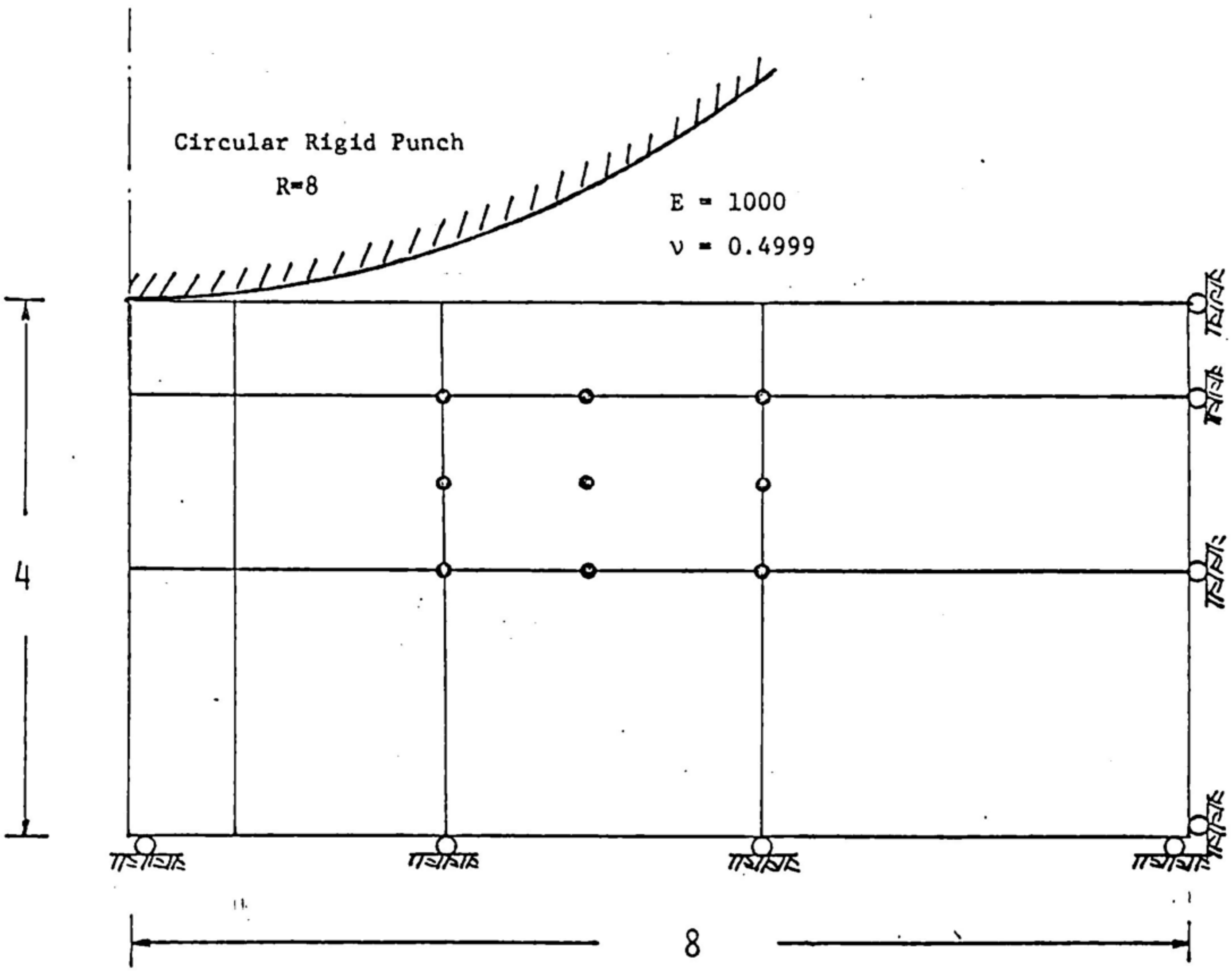

FIGURE 5. Finite Element Model of a Rigid Punch Problem.

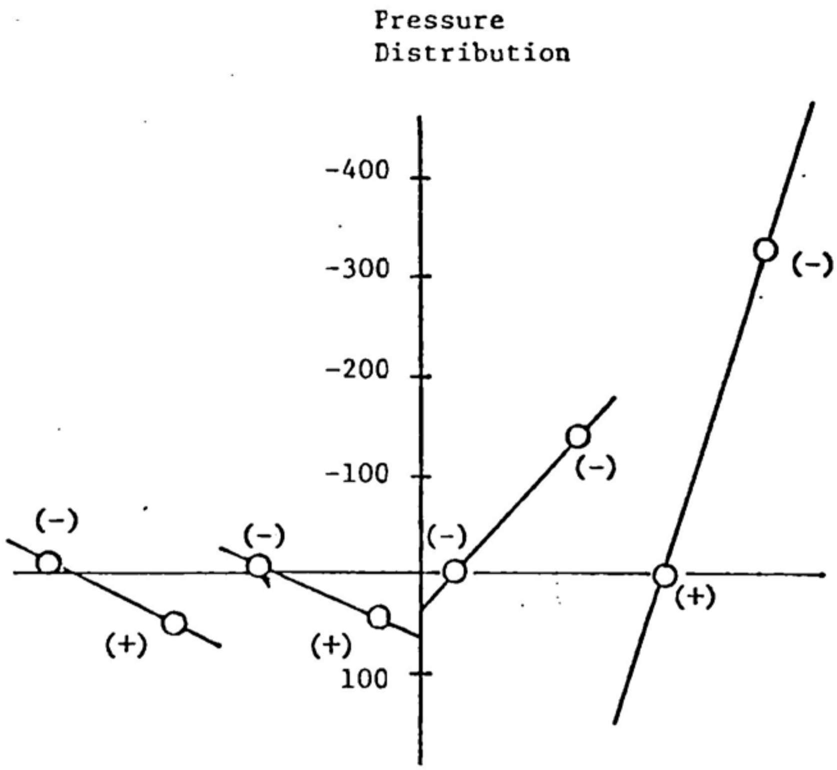

FIGURE 4. Pressure Distribution on the Cross Section A-A by 2 × 2 Gaussian Rule for a Singular Applied Body Force.