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Handbook of Numerical Analysis

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General Preface

During the past decades, giant needs for ever more sophisticated mathematical models and increasingly complex and extensive computer simulations have arisen. In this fashion, two indissociable activities, *mathematical modeling* and *computer simulation*, have gained a major status in all aspects of science, technology, and industry.

In order that these two sciences be established on the safest possible grounds, mathematical rigor is indispensable. For this reason, two companion sciences, *Numerical Analysis* and *Scientific Software*, have emerged as essential steps for validating the mathematical models and the computer simulations that are based on them.

Numerical Analysis is here understood as the part of *Mathematics* that describes and analyzes all the numerical schemes that are used on computers; its objective consists in obtaining a clear, precise, and faithful, representation of all the "information" contained in a mathematical model; as such, it is the natural extension of more classical tools, such as analytic solutions, special transforms, functional analysis, as well as stability and asymptotic analysis.

The various volumes comprising the *Handbook of Numerical Analysis* will thoroughly cover all the major aspects of Numerical Analysis, by presenting accessible and in-depth surveys, which include the most recent trends.

More precisely, the Handbook will cover the *basic methods of Numerical Analysis*, gathered under the following general headings:

- Solution of Equations in $\mathbb{R}^n$,
- Finite Difference Methods,
- Finite Element Methods,
- Techniques of Scientific Computing,
- Optimization Theory and Systems Science.

It will also cover the *numerical solution of actual problems of contemporary interest in Applied Mathematics*, gathered under the following general headings:

- Numerical Methods for Fluids,
- Numerical Methods for Solids,
- Specific Applications.

"Specific Applications" include: Meteorology, Seismology, Petroleum Mechanics, Celestial Mechanics, etc.

Each heading is covered by several *articles*, each of which being devoted to a specialized, but to some extent "independent", topic. Each article contains a thorough description and a mathematical analysis of the various methods in actual use, whose practical performances may be illustrated by significant numerical examples.

Since the Handbook is basically expository in nature, only the most basic results are usually proved in detail, while less important, or technical, results may be only stated or commented upon (in which case specific references for their proofs are systematically provided). In the same spirit, only a "selective" bibliography is appended whenever the roughest counts indicate that the reference list of an article should comprise several thousands items if it were to be exhaustive.

Volumes are numbered by capital Roman numerals (as Vol. I, Vol. II, etc.), according to their *chronological appearance*.

Since all the articles pertaining to a given *heading* may not be simultaneously available at a given time, a given heading usually appears in more than one volume; for instance, if articles devoted to the heading "Solution of Equations in $\mathbb{R}^n$" appear in Volumes I and III, these volumes will include "Solution of Equations in $\mathbb{R}^n$ (Part 1)" and "Solution of Equations in $\mathbb{R}^n$ (Part 2)" in their respective titles. Naturally, all the headings dealt with within a given volume appear in its title; for instance, the complete title of Volume I is "Finite Difference Methods (Part 1)—Solution of Equations in $\mathbb{R}^n$ (Part 1)".

Each article is subdivided into *sections*, which are numbered consecutively throughout the article by *Arabic numerals*, as Section 1, Section 2, ..., Section 14, etc. Within a given section, *formulas, theorems, remarks, and figures*, have their own independent numberings; for instance, within Section 14, formulas are numbered consecutively as (14.1), (14.2), etc., theorems are numbered consecutively as Theorem 14.1, Theorem 14.2, etc. For the sake of clarity, the article is also subdivided into *chapters*, numbered consecutively throughout the article by *capital Roman numerals;* for instance, Chapter I comprises Sections 1 to 9, Chapter II comprises Sections 10 to 16, etc.

P.G. Ciarlet
J.L. Lions
May 1989

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Finite Element Methods (Part 1)

Finite Elements: An Introduction

J. Tinsley Oden

Finite elements; perhaps no other family of approximation methods has had a greater impact on the theory and practice of numerical methods during the twentieth century. Finite element methods have now been used in virtually every conceivable area of engineering that can make use of models of nature characterized by partial differential equations. There are dozens of textbooks, monographs, handbooks, memoirs, and journals devoted to its further study; numerous conferences, symposia, and workshops on various aspects of finite element methodology are held regularly throughout the world. There exist easily over one hundred thousand references on finite elements today, and this number is growing exponentially with further revelations of the power and versatility of the method. Today, finite element methodology is making significant inroads into fields in which many thought were outside its realm; for example, computational fluid dynamics. In time, finite element methods may assume a position in this area of comparable or greater importance than classical difference schemes which have long dominated the subject.

Why finite elements?

A natural question that one may ask is: why have finite element methods been so popular in both the engineering and mathematical community? There is also the question, do finite element methods possess properties that will continue to make them attractive choices of methods to solve difficult problems in physics and engineering?

In answering these questions, one must first point to the fact that finite element methods are based on the weak, variational, formulation of boundary and initial value problems. This is a critical property, not only because it provides a proper setting for the existence of very irregular solutions to differential equations (e.g. distributions), but also because the solution appears in the integral of a quantity over

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a domain. The simple fact that the integral of a measurable function over an arbitrary domain can be broken up into the sum of integrals over an arbitrary collection of almost disjoint subdomains whose union is the original domain, is a vital observation in finite element theory. Because of it, the analysis of a problem can literally be made locally, over a typical subdomain, and by making the subdomain sufficiently small one can argue that polynomial functions of various degrees are adequate for representing the local behavior of the solution. This summability of integrals is exploited in every finite element program. It allows the analysts to focus their attention on a typical finite element domain and to develop an approximation independent of the ultimate location of that element in the final mesh.

The simple integral property also has important implications in physics and in most problems in continuum mechanics. Indeed, the classical balance laws of mechanics are global, in the sense that they are integral laws applying to a given mass of material, a fluid or solid. From the onset, only regularity of the primitive variables sufficient for these global conservation laws to make sense is needed. Moreover, since these laws are supposed to be fundamental axioms of physics, they must hold over every finite portion of the material: every finite element of the continuum. Thus once again, one is encouraged to think of approximate methods defined by integral formulations over typical pieces of a continuum to be studied.

These rather primitive properties of finite elements lead to some of its most important features:

(1) *Arbitrary geometries.* The method is essentially geometry-free. In principle, finite element methods can be applied to domains of arbitrary shape and with quite arbitrary boundary conditions.

(2) *Unstructured meshes.* While there is still much prejudice in the numerical analysis literature toward the use of coordinate-dependent algorithms and mesh generators, there is nothing intrinsic in finite element methodology that requires such devices. Indeed, finite element methods by their nature lead to unstructured meshes. This means, in principle, analysts can place finite elements anywhere they please. They may thus model the most complex types of geometries in nature and physics, ranging from the complex cross-sections of biological tissues to the exterior of aircraft to internal flows in turbo machinery, without strong use of a global fixed coordinate frame.

(3) *Robustness.* It is well known that in finite element methods the contributions of local approximations over individual elements are assembled together in a systematic way to arrive at a global approximation of a solution to a partial differential equation. Generally, this leads to schemes which are stable in appropriate norms, and, moreover, insensitive to singularities or distortions of the mesh, in sharp contrast to classical difference methods. There are notable exceptions to this, of course, and these exceptions have been the subject of some of the most important works in finite element theory. But, by and large, the direct use of Galerkin or Petrov–Galerkin methods to derive finite element methods leads to conservative and stable algorithms, for most classes of problems in mechanics and mathematical physics.

(4) *Mathematical foundation.* Because of the extensive work on the mathematical foundations done during the seventies and eighties, finite elements now enjoy a rich and solid mathematical basis. The availability of methods to determine a priori and a posteriori estimates provides a vital part of the theory of finite elements, and makes it possible to lift the analysis of important engineering and physical problems above the traditional empiricism prevalent in many numerical and experimental studies.

These properties are intrinsic to finite element methods and continue to make these methods among the most attractive for solving complex problems.

They represent the most desirable properties of any numerical scheme designed to handle real-world problems. Moreover, the basic features of finite element methodology provide an ideal setting for innovative use of modern supercomputing architectures, particularly parallel processing. For these reasons, it is certain that finite element concepts will continue to occupy an important role in applications and in research on the numerical solution of partial differential equations.

The early history

When did finite elements begin? It is difficult to trace the origins of finite element methods because of a basic problem in defining precisely what constitutes a "finite element method". To most mathematicians, it is a method of piecewise polynomial approximation and, therefore, its origins are frequently traced to the appendix of a paper by COURANT [1943] in which piecewise linear approximations of the Dirichlet problem over a network of triangles is discussed. Also, the "interpretation of finite differences" by PÓLYA [1952] is regarded as embodying piecewise polynomial approximation aspects of finite elements.

On the other hand, the approximation of variational problems on a mesh of triangles goes back much further: 92 years. In 1851, SCHELLBACH [1851] proposed a finite-element-like solution to Plateau's problem of determining the surface S of minimum area enclosed by a given closed curve. Schellbach used an approximation S_h of S by a mesh of triangles over which the surface was represented by piecewise linear functions, and he then obtained an approximation to the solution to Plateau's problem by minimizing S_h with respect to the coordinates of hexagons formed by six elements (see WILLIAMSON [1980]). Not quite the conventional finite element approach, but certainly as much a finite element technique as that of Courant.

Some say that there is even an earlier work that uses some of the ideas underlying finite element methods: Gottfried Leibniz himself employed a piecewise linear approximation of the Brachistochrone problem proposed by Johann Bernoulli in 1696 (see the historical volume, LEIBNIZ [1962]). With the help of his newly developed calculus tools, Leibniz derived the governing differential equation for the problem, the solution of which is a cycloid. However, most would agree that to credit this work as a finite element approximation is somewhat stretching the point. Leibniz had no intention of approximating a differential equation; rather, his purpose was to derive one. Two and a half centuries later it was realized that useful approximations of differential equations could be determined by not necessarily taking infinitesimal elements as in the calculus, but by keeping the elements finite in size. This idea is, in fact, the basis of the term "finite element".

There is also some difference in the process of laying a mesh of triangles over a domain on the one hand and generating the domain of approximation by piecing together triangles on the other. While these processes may look the same in some cases, they may differ dramatically in how the boundary conditions are imposed. Thus, neither Schellbach nor Courant, nor for that matter Synge who used triangular meshes many years later, were particularly careful as to how boundary conditions were to be imposed or as to how the boundary of the domain was to be modeled by elements, issues that are now recognized as an important feature of finite element methodologies. If a finite element method is one in which a global approximation of a partial differential equation is built up from a sequence of local approximations over subdomains, then credit must go back to the early papers of HRENNIKOFF [1941], and perhaps beyond, who chose to solve plane elasticity problems by breaking up the domain of the displacements into little finite pieces, over which the stiffnesses were approximated using bars, beams, and spring elements. A similar "lattice analogy" was used by McHENRY [1943]. While these works are draped in the most primitive physical terms, it is nevertheless clear that the methods involve some sort of crude piecewise linear or piecewise cubic approximation over rectangular cells. Miraculously, the methods also seem to be convergent.

To the average practitioner who uses them, finite elements are much more than a method of piecewise polynomial approximation. The whole process of partitioning of domains, assembling elements, applying loads and boundary conditions, and, of course, along with it, local polynomial approximation, are all components of the finite element method.

If this is so, then one must acknowledge the early papers of Gabriel Kron who developed his "tensor analysis of networks" in 1939 and applied his "method of tearing" and "network analysis" to the generation of global systems from large numbers of individual components in the 1940s and 1950s (KRON [1939]; see also KRON [1953]). Of course, Kron never necessarily regarded his method as one of approximating partial differential equations; rather, the properties of each component were regarded as exactly specified, and the issue was an algebraic one of connecting them all appropriately together.

In the early 1950s, ARGYRIS [1954] began to put these ideas together into what some call a primitive finite element method: he extended and generalized the combinatoric methods of Kron and other ideas that were being developed in the literature on system theory at the time, and added to it variational methods of approximation, a fundamental step toward true finite element methodology.

Around the same time, SYNGE [1957] described his "method of the hypercircle" in which he also spoke of piecewise linear approximations on triangular meshes, but not in a rich variational setting and not in a way in which approximations were built by either partitioning a domain into triangles or assembling triangles to approximate a domain (indeed Synge's treatment of boundary conditions was clearly not in the spirit of finite elements, even though he was keenly aware of the importance of convergence criteria and of the "angle condition" for triangles, later studied in some depth by others).

It must be noted that during the mid-1950s there were a number of independent studies underway which made use of "matrix methods" for the analysis of aircraft structures. A principal contributor to this methodology was LEVY [1953] who introduced the "direct stiffness method" wherein he approximated the structural behavior of aircraft wings using assemblies of box beams, torsion boxes, rods and shear panels. These assuredly represent some sort of crude local polynomial approximation in the same spirit as the Hrennikoff and McHenry approaches. The direct stiffness method of Levy had a great impact on the structural analysis of aircraft, and aircraft companies throughout the United States began to adopt and apply some variant of this method or of the methods of Argyris to complex aircraft structural analyses. During this same period, similar structural analysis methods were being developed and used in Europe, particularly in England, and one must mention in this regard the work of TAIG [1961] in which shear lag in aircraft wing panels was approximated using basically a bilinear finite element method of approximation. Similar element-like approximations were used in many aircraft industries as components in various matrix methods of structural analyses. Thus the precedent was established for piecewise approximations of some kind by the mid-1950s.

To a large segment of the engineering community, the work representing the beginning of finite elements was that contained in the pioneering paper of TURNER, CLOUGH, MARTIN and TOPP [1956] in which a genuine attempt was made at both a local approximation (of the partial differential equations of linear elasticity) and the use of assembly strategies essential to finite element methodology. It is interesting that in this paper local element properties were derived without the use of variational principles. It was not until 1960 that CLOUGH [1960] actually dubbed these techniques as "finite element methods" in a landmark paper on the analysis of linear plane elasticity problems.

The 1960s were the formative years of finite element methods. Once it was perceived by the engineering community that useful finite element methods could be derived from variational principles, variationally based methods significantly dominated all the literature for almost a decade. If an operator was unsymmetric, it was thought that the solution of the associated problem was beyond the scope of finite elements, since it did not lend itself to a traditional extremum variational approximation in the spirit of Rayleigh and Ritz.

From 1960 to 1965, a variety of finite element methods were proposed. Many were primitive and unorthodox; some were innovative and successful. During this time, a variety of attempts at solving the biharmonic equation for plate bending problems were proposed which employed piecewise polynomial approximations, but did not provide the essentials for convergence. This led to the concern of some as to whether the method was indeed applicable to such problems. On the other hand, it was clear that classical Fourier series solutions of plate problems were, under appropriate conditions, convergent and could be fit together in an assemblage of rectangular components (ODEN [1962]) and, thus, a form of "spectral finite element methods" was introduced early in the study of such problems. However, such high-order schemes never received serious attention in this period, as it was felt that piecewise polynomial approximations could be developed which did give satisfactory results. It was not until the mid- to late 1960s that papers on bicubic spline approximations by BOGNER, FOX, and SCHMIT [1966] and BIRKHOFF, SCHULTZ, and VARGA [1968] provided successful polynomial finite element approximations for these classes of problems.

Many workers in the field feel that the famous Dayton conferences on finite elements (at the Air Force Flight Dynamics Laboratory in Dayton, Ohio, USA) represented landmarks in the development of the field (see PRZEMIENIECKI et al. [1966]). Held in 1965, 1968, 1970, these meetings brought specialists from all over the world to discuss their latest triumphs and failures, and the pages of the proceedings, particularly the earlier volumes, were filled with remarkable and innovative accomplishments from a technical community just beginning to learn the richness and power of this new collection of ideas. In these volumes one can find many of the premier papers of now well-known methods. In the first volume alone one can find mixed finite element methods (HERRMANN [1966]), Hermite approximations (PESTEL [1966]), C^1-bicubic approximations (BOGNER, FOX and SCHMIT [1966]), hybrid methods (PIAN [1966]) and other contributions. In later volumes, further assaults on nonlinear problems and special element formulations can be found.

Near the end of the sixties and early seventies there finally emerged the realization that the method could be applied to unsymmetric operators without difficulty and thus problems in fluid mechanics were brought within the realm of application of finite element methods; in particular, finite element models of the full Navier–Stokes equations were first presented during this period (ODEN [1969], ODEN and SOMOGYI [1968], ODEN [1970]).

The early textbook by ZIENKIEWICZ and CHEUNG [1967] did much to popularize the method with the practicing engineering community. However, the most important factor leading to the rise in popularity during the late 1960s and early 1970s was not purely the publication of special formulations and algorithms, but the fact that the method was being very successfully used to solve difficult engineering problems. Much of the technology used during this period was due to Bruce Irons, who with his colleagues and students developed a multitude of techniques for the successful implementation of finite elements. These included the frontal solution technique (IRONS [1970]), the patch test (IRONS and RAZZAQUE [1972]), isoparametric elements (ERGATOUDIS, IRONS and ZIENKIEWICZ [1966]), and numerical integration schemes (IRONS [1966]) and many more. The scope of finite element applications in the 1970s would have been significantly diminished without these contributions.

The mathematical theory

The mathematical theory of finite elements was slow to emerge from this caldron of activity. The beginning works on the mathematical theory of finite elements were understandably concerned with one-dimensional elliptic problems and used many of the tools and jargon of Ritz methods, interpolation, and variational differences. An early work in this line was the paper of VARGA [1966] which dealt with "Hermite interpolation-type Ritz methods" for two-point boundary value problems. We also mention in this regard the paper of BIRKHOFF, DE BOOR, SCHWARTZ and WENDROFF [1966] on "Rayleigh–Ritz approximation by piecewise cubic polynomials". This is certainly one of the first papers to deal with the issue of convergence of finite element methods, although some papers on variational differences yielded similar results but did not focus on the piecewise polynomial features of finite elements. The work of KANG FENG [1965], published in Chinese (a copy of which I have not been able to acquire for review) may fall into this category and is sometimes noted as relevant to the convergence of finite element methods.

The mathematical theory of finite elements for two-dimensional and higher-dimensional problems began in 1968 and several papers were published that year on the subject. One of the first papers in this period to address the problem of convergence of a finite method in a rigorous way and in which a priori error estimates for bilinear approximations of a problem in plane elasticity are obtained, is the often overlooked paper of JOHNSON and MCLAY [1968], which appeared in the *Journal of Applied Mechanics*. This paper correctly developed error estimates in energy norms, and even attempted to characterize the deterioration of convergence rates due to corner singularities. In the same year there appeared the first of two important papers by OGENESJAN and RUCHOVEC [1968, 1969] in the Russian literature, in which "variational difference schemes" were proposed for linear second-order elliptic problems in two-dimensional domains. These works dealt with the estimates of the rate of convergence of variational difference schemes.

Also in 1968 there appeared the important mathematical paper of ZLÁMAL [1968] in which a detailed analysis of interpolation properties of a class of triangular elements and their application to second-order and fourth-order linear elliptic boundary value problems is discussed. This paper attracted the interest of a large segment of the numerical analysis community and several very good mathematicians began to work on finite element methodologies. The paper by Zlámal also stands apart from other multidimensional finite element papers of this era since it represented a departure of studies of tensor products of polynomials on rectangular domains and provided an approach toward approximations in general polygonal domains. In the same year, CIARLET [1968] published a rigorous proof of convergence of piecewise linear finite element approximation of a class of linear two-point boundary value problems and proved L^∞ estimates using a discrete maximum principle. We also mention the work of OLIVEIRA [1968] on convergence of finite element methods which established correct rates of convergence for certain problems in appropriate energy norms.

A year later, SCHULTZ [1969] presented error estimates for "Rayleigh–Ritz–Galerkin methods" for multidimensional problems. Two years later, SCHULTZ [1971] published L^2 error bounds for these types of methods.

By 1972, finite element methods had emerged as an important new area of numerical analysis in applied mathematics. Mathematical conferences were held on the subject on a regular basis, and there began to appear a rich volume of literature on mathematical aspects of the method applied to elliptic problems, eigenvalue problems, and parabolic problems. A conference of special significance in this period was held at the University of Maryland in 1972 and featured a penetrating series of lectures by Ivo Babuška (see BABUŠKA and AZIZ [1972]) and several important mathematical papers by leading specialists in the mathematics of finite elements, all collected in the volume edited by AZIZ [1972].

One unfamiliar with aspects of the history of finite elements may be led to the erroneous conclusion that the method of finite elements emerged from the growing wealth of information on partial differential equations, weak solutions of boundary value problems, Sobolev spaces, and the associated approximation theory for elliptic variational boundary value problems. This is a natural mistake, because the seeds for the modern theory of partial differential equations were sown about the same time as those for the development of modern finite element methods, but in an entirely different garden.

In the late 1940s, Laurent Schwartz was putting together his theory of distributions around a decade after the notion of generalized functions and their use in partial differential equations appeared in the pioneering work of S.L. Sobolev. A long list of other names could be added to the list of contributors to the modern theory of partial differential equations, but that is not our purpose here. Rather, we must only note that the rich mathematical theory of partial differential equations which began in the 1940s and 1950s, blossomed in the 1960s, and is now an integral part of the foundations of not only partial differential equations but also approximation theory, grew independently and parallel to the development of finite element methods for almost two decades. There was important work during this period on the foundations of variational methods of approximation, typified by the early work of LIONS [1955] and by the French school in the early 1960s; but, while this work did concern itself with the systematic development of mathematical results that would ultimately prove to be vital to the development of finite element methods, it did not focus on the specific aspects of existing and already successful finite element concepts. It was, perhaps, an unavoidable occurrence, that in the late 1960s these two independent subjects, finite element methodology and the theory of approximation of partial differential equations via functional analysis methods, united in an inseparable way, so much so that it is difficult to appreciate the fact that they were ever separate.

The 1970s must mark the decade of the mathematics of finite elements. During this period, great strides were made in determining a priori error estimates for a variety of finite element methods, for linear elliptic boundary value problems, for eigenvalue problems, and certain classes of linear and nonlinear parabolic problems; also, some preliminary work on finite element applications to hyperbolic equations was done. It is both inappropriate and perhaps impossible to provide an adequate survey of this large volume of literature, but it is possible to present an albeit biased reference to some of the major works along the way.

An important component in the theory of finite elements is an interpolation theory: how well can a given finite element method approximate functions of a given class locally over a typical finite element? A great deal was known about this subject from the literature on approximation theory and spline analysis, but its particularization to finite elements involves technical difficulties. One can find results on finite element interpolation in a number of early papers, including those of ZLÁMAL [1968], BIRKHOFF [1969], SCHULTZ [1969], BRAMBLE and ZLÁMAL [1970], BABUŠKA [1970, 1971], and BABUŠKA and AZIZ [1972]. But the elegant work on Lagrange and Hermite interpolations of finite elements by CIARLET and RAVIART [1972a] must stand as a very important contribution to this vital aspect of finite element theory.

A landmark work on the mathematics of finite elements appeared in 1972 in the remarkably comprehensive and penetrating memoir of BABUŠKA and AZIZ [1972] on the mathematical foundations of finite element methods. Here one can find interwoven with the theory of Sobolev spaces and elliptic problems, general results on approximation theory that have direct bearing on finite element methods. It was known that Cea's lemma (CEA [1964]) established that the approximation error in a Galerkin approximation of a variational boundary value problem is bounded by the so-called interpolation error; that is, the distance in an appropriate energy norm from the solution of the problem to the subspace of approximations. Indeed, it was this fact that made the results on interpolation theory using piecewise polynomials of particular interest in finite element methods. In the work of BABUŠKA [1971] and BABUŠKA and AZIZ [1972], this framework was dramatically enlarged by Babuška's introduction of the so-called "INF-SUP" condition. This condition is encountered in the characterization of coerciveness of bilinear forms occuring in elliptic boundary value problems. The characterization of this "INF-SUP" condition for the discrete finite element approximation embodies in it the essential elements for studying the stability in convergence of finite element methods. BREZZI [1974] developed an equivalent condition for studying constrained elliptic problems and these conditions provide for a unified approach to the study of qualitative properties, including rates of convergence, of broad classes of finite element methods.

The fundamental work of NITSCHE [1970] on L^∞ estimates for general classes of linear elliptic problems must stand out as one of the most important contributions of the seventies. STRANG [1972], in an important communication, pointed out "variational crimes", inherent in many finite element methods, such as improper numerical quadrature, the use of nonconforming elements, improper satisfaction of boundary conditions, etc., all common practices in applications, but all frequently leading to exceptable numerical schemes.

In the same year, CIARLET and RAVIART [1972b, c] also contributed penetrating studies of these issues. Many of the advances of the 1970s drew upon earlier results on variational methods of approximation based on the Ritz method and finite differences; for example the fundamental Aubin–Nitsche method for lifting the order of convergence to lower Sobolev norms (see AUBIN [1967] and NITSCHE [1963]; see also OGENESJAN and RUCHOVEC [1969]) used such results. In 1974, the important paper of BREZZI [1974] mentioned earlier, used such earlier results on saddle point problems and laid the groundwork for a multitude of papers on problems with constraints and on the stability of various finite element procedures. While convergence of special types of finite element strategies such as mixed methods and hybrid methods had been attempted in the early 1970s (e.g. ODEN [1972]), the Brezzi results, and the methods of Babuška for constrained problems, provided a general framework for studying virtually all mixed and hybrid finite elements (e.g. RAVIART [1975], RAVIART and THOMAS [1977], BABUŠKA, ODEN and LEE [1977]).

The first textbook on mathematical properties of finite element methods was the popular book of STRANG and FIX [1973]. A book on an introduction to the mathematical theory of finite elements was published soon after by ODEN and REDDY [1976] and the well-known treatise on the finite element method for elliptic problems by CIARLET [1978] appeared two years later.

The penetrating work of NITSCHE and SCHATZ [1974] on interior estimates and SCHATZ and WAHLBIN [1978] on L^∞ estimates and singular problems represented notable contributions to the growing mathematical theory of finite elements. The important work of DOUGLAS and DUPONT (e.g. [1970, 1973]; DUPONT [1973]) on finite element methods for parabolic problems and hyperbolic problems must be mentioned along with the idea of elliptic projections of WHEELER [1973] which provided a useful technique for deriving error bounds for time-dependent problems.

The 1970s also represented a decade in which the generality of finite element methods began to be appreciated over a large portion of the mathematics and scientific community, and it was during this period that significant applications to highly nonlinear problems were made. The fact that very general nonlinear phenomena in continuum mechanics, including problems of finite deformation of solids and of flow of viscous fluids could be modeled by finite elements and solved on existing computers was demonstrated in the early seventies (e.g. ODEN [1972]), and, by the end of that decade, several "general purpose" finite element programs were in use by engineers to treat broad classes of nonlinear problems in solid mechanics and heat transfer. The mathematical theory for nonlinear problems also was advanced in this period, and the important work of FALK [1974] on finite element approximations of variational inequalities should be mentioned.

It is not too inaccurate to say that by 1980, a solid foundation for the mathematical theory of finite elements for linear problems had been established and that significant advances in both theory and application into nonlinear problems existed. The open questions that remain are difficult ones and their solution will require a good understanding of the mathematical properties of the method. The works collected in this volume should not only provide a summary of important results and approaches to mathematical issues related to finite elements, but also they should provide a useful starting point for further research.

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